JHEP HYPER VERSION

Received: *October 12, 2001*, Accepted: *December 3, 2001*

Toric duality is Seiberg duality

Chris E. Beasley

Joseph Henry Laboratories, Princeton University
Princeton, New Jersey 08544, USA
E-mail: `cbeasley@princeton.edu`

M. Ronen Plesser

Center for Geometry and Theoretical Physics
Box 90318, Duke University, Durham, NC 27708-0318, USA
E-mail: `plesser@cgtp.duke.edu`

Abstract: We study four $\mathcal{N} = 1$ $\mathrm{SU}(N)^6$ gauge theories, with bi-fundamental chiral matter and a superpotential. In the infrared, these gauge theories all realize the low-energy world-volume description of N coincident D3-branes transverse to the complex cone over a del Pezzo surface dP_3 which is the blowup of $\mathbb{P}^2$ at three generic points. Therefore, the four gauge theories are expected to fall into the same universality class — an example of a phenomenon that has been termed "toric duality". However, little independent evidence has been given that such theories are infrared-equivalent. In fact, we show that the four gauge theories are related by the $\mathcal{N} = 1$ duality of Seiberg, vindicating this expectation. We also study holographic aspects of these gauge theories. In particular we relate the spectrum of chiral operators in the gauge theories to wrapped D3-brane states in the AdS dual description. We finally demonstrate that the other known examples of toric duality are related by $\mathcal{N} = 1$ duality, a fact which we conjecture holds generally.

Keywords: Duality in Gauge Field Theories, Brane Dynamics in Gauge Theories, AdS-CFT Correspondance, Supersymmetry and Duality.

JHEP12(2001)001

Contents

JHEP12(2001)001

1. Introduction

The low-energy worldvolume dynamics of a system of N D3-branes is described by an $\mathcal{N} = 4$ supersymmetric $\mathrm{U}(N)$ Yang-Mills theory. This has a holographic dual – a compactification of the type-IIB string on $AdS_5 \times S^5$ [1]. At extremely low energies the worldvolume theory depends on the closed-string backgrounds only through their values on the branes. More interesting worldvolume theories can be obtained by studying sets of intersecting branes or by locating the branes near a singularity of the transverse space, and understanding these theories is of considerable interest. For example, they shed light on the way in which branes probe spacetime [2], as well as providing local models for studying the spectrum of brane states in nontrivial backgrounds [3].

If the singularity in question can be realized by partially resolving a quotient singularity, then an array of machinery exists to derive the worldvolume gauge theory governing the low-energy dynamics. The theories arising at quotient singularities were first described in [4]. To describe N branes near a singularity of the form $\mathbb{C}^n/\Gamma$ one begins with $|\Gamma|N$ branes on the covering space, lifting the action of the discrete group to the Chan-Paton indices using the regular representation. The worldvolume theory on the quotient is given by a projection to Γ-invariant degrees of freedom. One finds a theory with a product gauge group, with factors associated to the irreducible representations of Γ, a supersymetry determined by the Γ-invariant spinors of $\mathrm{SO}(2n)$, and matter in various representations. In the cases of interest here, $n = 3$, $\Gamma \subset \mathrm{SU}(3)$ preserves four supercharges, the factors in the unbroken gauge group will be $\mathrm{U}(N)$ groups, and the matter fields will be chiral multiplets in bifundamental representations. The closed-string background determines the parameters of the theory. In particular, the metric deformations corresponding to resolutions of the singularity determine Fayet-Iliopoulos terms in the gauge theory.[1] The moduli space of vacua of the gauge theory should describe (at least) the positions of the branes in the transverse space and so contain its N-th symmetric power.

In [6] it was noted that for $N = 1$ this procedure yields a moduli space of vacua described naturally as a toric variety.[2] This idea was used in [5] to systematically discuss the low-energy dynamics of branes near any toric singularity, and to study an extension of the holographic conjectures to singular transverse spaces. The method used realizes the singularity of interest as a partial resolution of a $\mathbb{Z}_n \times \mathbb{Z}_m$ quotient. The description of the quotient theory and the map of metric deformations to FI parameters is then used to find a theory describing the partially resolved space. In essence the FI terms lead to a nonzero expectation value for some charged fields, leading to spontaneous breaking of the gauge symmetry. The extreme low-energy behavior near an appropriately chosen point in the moduli space of vacua of this theory then describes the dynamics of the branes near the singularity in question. In particular, the low-energy limit is completely independent of any global properties of the transverse space, as usual. By choosing the scale of the blowups (FI terms) appropriately, it is possible to arrange a situation in which the dynamics below this scale is determined to a good approximation by the quotient gauge theory (string corrections are suppressed).

For a weakly-coupled string theory the dynamics of the theory at the FI scale is perturbative and the gauge symmetry breaking can be described by a classical Higgs calculation. The resulting gauge theories are in general not conformal. The expectation is that the extreme low-energy behavior of these theories will be described by a nontrivial RG fixed

[1] We are being imprecise here. In fact, the worldvolume gauge group is a product of $\mathrm{SU}(N)$ factors for reasons discussed below, and does not admit FI terms. The relevant twisted closed-string modes correspond to expectation values of "baryonic" degrees of freedom in the worldvolume theory. The description given here is nonetheless a useful way of computing the moduli space of vacua [5], as we explain.

[2] This is not as obvious as it sounds; that the moduli space of vacua of an $\mathcal{N} = 1$ supersymmetric abelian gauge theory is toric is clear. There are two subtleties here, however, and these are important for our study. First, the gauge theory corresponding to the quotient features a cubic superpotential (descended from the interaction in the $\mathcal{N} = 4$ theory corresponding to the covering space); the moduli space of vacua is thus obtained naturally as a subspace of a toric variety given by the critical points of this. The insight of [6] is that this space is itself a toric variety. A further subtlety, physical rather than technical, is that the gauge group factors are in fact of the form $\mathrm{SU}(N)$ rather than $\mathrm{U}(N)$. The spaces we will describe are thus not the full moduli space of vacua but in fact describe a piece of this associated to motion of the branes in the transverse space.

JHEP12(2001)001

point. For generic values of the FI terms, of course, this will simply be the $\mathcal{N} = 4$ theory describing branes at a smooth point. At the special values for which we obtain a particular singular space we should find new conformal theories. The theory obtained in this way corresponding to a conical singularity (for which the cone $C(H)$ over a five-manifold H — the "non-spherical horizon" — is a linear approximation) is expected to be holographically dual to a compactification of the type-IIB string on $AdS_5 \times H$ [5]. The $\mathcal{N} = 2$ unbroken supersymmetry of this compactification implies that H be an Einstein-Sasaki space [7]. In particular, it must have a U(1) isometry corresponding to the R-symmetry of the worldvolume theory. When the action is regular, the quotient X is a Kähler-Einstein manifold of positive curvature, and H is recovered as the unit circle bundle in a hermitean line bundle over X. The cone $C(H)$ is then the complex cone over X.

Toric methods are thus a powerful tool for probing the dynamics of D3-branes at orbifold singularities and partial resolutions thereof. When the methods of [5] were extended to a more complicated example in [8] it was found that in general there is a puzzling ambiguity. The translation of metric deformations to FI terms is far from unique. Typically, one finds many, many separate cones in the space of Fayet-Iliopoulos terms which all correspond to the same geometric phase. This was systematically studied in [9, 10]. In general, following the procedure outlined above in the different cones will lead to *distinct* gauge theories describing the low-energy dynamics. We stated above that the extreme low-energy behavior is determined completely by the local structure of the singularity, and the multiplicity of models thus poses a puzzle. The authors of [10] conjectured that in fact the various models must all flow to the same infrared conformal fixed point determined by the singularity, and termed the relation between them "toric duality". There is not, however, an independent verification of the statement.

In this paper, we study some examples of toric dualities, including the examples of [10]. We find that in all these examples, the various gauge theories corresponding to a given singularity are related by a version of Seiberg duality [11], providing a relation between them intrinsic to the gauge theory. This is believed to be an exact statement relating pairs of SQCD theories. In our models, the SQCD in question is deformed by coupling to other degrees of freedom in the theory, and when we say two models are related by Seiberg duality we mean they are given by dual deformations of the two dual SQCD theories.

In general, the gauge theories we find are not asymptotically free. This means we cannot usefully think of the RG trajectory we follow to the conformal fixed point in terms of the weakly coupled theory. Technically, the gauge couplings for these factors are an example of "dangerously irrelevant" deformations. In fact, even when our models are asymptotically free, the superpotentials we find contain such operators. Theories containing such operators were studied in a related context in [12], and our models exhibit many of the subtleties found there. For instance, we find that a perturbative calculation of anomaly coefficients in each of the four models does not satisfy the anomaly-matching criterion of 't Hooft. We also find that the classical rings of chiral operators in the four models are truncated by additional quantum relations in the infrared, a statement which we motivate both from Seiberg duality and the AdS/CFT correspondence.

JHEP12(2001)001

We will focus here on understanding a particular example of toric duality and will indicate the extension to the other known examples. The example of interest arises from the partial resolution of $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ which is locally a complex cone over a del Pezzo surface $X = dP_3$ which can be realized as a blowup of $\mathbb{P}^2$ at three generic points. X is known to possess a smooth Kähler-Einstein metric [13, 14] and the corresponding horizon H is Einstein-Sasaki with regular U(1)-action [5]. In this case, there are in fact 1,602 cones in the space of Fayet-Iliopoulos parameters corresponding to the partial resolution which leads to this singularity. Values of the FI parameters in each of these cones lead to one of four distinct $\mathcal{N} = 1$ $\mathrm{SU}(N)^6$ gauge-theories whose infrared dynamics describe the low-energy excitations of N coincident D3-branes transverse to this singularity.[3] The presence of the $\mathcal{N} = 1$ supersymmetry makes the study of the infrared behaviour of these gauge theories a tractable problem.

In section 2, we first sketch the $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ orbifold gauge theory and derive from it the four gauge theories describing the complex cone over X. We study the four theories in some detail, finding evidence to support the conjecture that they flow to a common fixed point. We then demonstrate that the four gauge theories are related to each other by a sequence of $\mathcal{N} = 1$ dualities.

In section 3, we review the classical algebraic geometry and topology of X and the cone C, and the differential geometry of the non-spherical horizon H associated to the dual AdS description. This analysis is very useful for understanding some general properties of the four gauge theories. It is necessary to understand and explain the spectrum of baryonic operators in the gauge theories via wrapped D3-brane states in the dual AdS theory. (Unfortunately, a rather detailed analysis of the geometry is required for this purpose, but we believe that the payoff for this effort in section 4 makes it worthwhile.)

In section 4, we apply the $\mathcal{N} = 1$ dualities to understand quantum corrections to the ring of chiral operators in the gauge theories. We use this determination of the algebra of chiral operators to describe the spectrum of D3-branes wrapping supersymmetric three-cycles on H in the dual AdS theory.

In section 5, we discuss the other known examples of toric duality [10] and show how they, too, can be understood using Seiberg duality. We conclude in section 6 with some ideas for further study.

In the appendices, we list some relevant facts about chiral operators in each of the four gauge theories.

2. The four dP_3 models

As a review, we first recall how the four gauge theories describing N D3-branes transverse to the cone over dP_3 arise from the $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ orbifold theory. The orbifold theory itself is

[3] As a sidenote, we remark that this fact casts doubt on the explanation offered in [15] for the toric duality, since the authors' explanation would predict a unique gauge theory for the case $X = dP_3$. In fact, the various global choices in how one resolves the orbifold (to which the authors attribute the existence of the distinct gauge theories) are related by the large quantum symmetry group of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold gauge theory and so cannot be responsible for the ambiguity.

JHEP12(2001)001

determined by lifting N coincident D3-branes at the origin of the orbifold to $9N$ coincident D3-branes on the cover. The worldvolume $\mathrm{U}(9N)$ super Yang-Mills theory on the covering branes is then projected onto the sector invariant under the $\mathbb{Z}_3 \times \mathbb{Z}_3$ action, lifted to the Chan-Paton bundle using the regular representation. Projecting to the invariant fields reduces the $\mathrm{U}(9N)$ gauge group to $\mathrm{U}(N)^9$. The $\mathrm{U}(1)$ factors (except the diagonal "center of mass" factor, which decouples) are broken by the coupling to twisted sector closed-string states [4]. The gauge group on the worldvolume is thus $\mathrm{SU}(N)^9$, while the $\mathrm{U}(1)^8$ remains a global symmetry of the theory. The orbifold preserves an $\mathcal{N}=1$ supersymmetry. The superpotential of the orbifold theory is the projection of the $\mathcal{N}=4$ superpotential $W = \mathrm{tr}([X,Y]Z)$ and hence is also cubic. All chiral matter transforms in bifundamental representations of the gauge group, and the representation content of the theory is neatly summarized by a quiver diagram (see [9] for this quiver).

We will also use quiver diagrams to summarize the representation contents of the models of interest, so we mention how these diagrams are read. In quiver diagrams, the nodes represent the semi-simple $\mathrm{SU}(N)$ factors of the $\mathrm{SU}(N)^p$ gauge-group, and an arrow from node i to node j indicates a chiral superfield which transforms as $\overline{\mathbf{N}}$ under $\mathrm{SU}(N)_i$, as $\mathbf{N}$ under $\mathrm{SU}(N)_j$, and as a singlet under the other factors of the gauge-group. Such a field we denote by X_{ji}.

In this notation, the matter content of the $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ orbifold theory consists of twenty-seven chiral superfields

$$\begin{aligned}
&X_{15}, X_{26}, X_{34}, X_{48}, X_{59}, X_{67}, X_{72}, X_{83}, X_{91},\\
&Y_{13}, Y_{21}, Y_{32}, Y_{46}, Y_{54}, Y_{65}, Y_{79}, Y_{87}, Y_{98},\\
&Z_{17}, Z_{28}, Z_{39}, Z_{41}, Z_{52}, Z_{63}, Z_{74}, Z_{85}, Z_{96},
\end{aligned} \tag{2.1}$$

and the orbifold superpotential is

$$\begin{aligned}
W = \mathrm{tr}\Big[&Z_{17}(X_{72}Y_{21} - Y_{79}X_{91}) + Z_{28}(X_{83}Y_{32} - Y_{87}X_{72}) + \\
&+ Z_{39}(X_{91}Y_{13} - Y_{98}X_{83}) + Z_{41}(X_{15}Y_{54} - Y_{13}X_{34}) + \\
&+ Z_{52}(X_{26}Y_{65} - Y_{21}X_{15}) + Z_{63}(X_{34}Y_{46} - Y_{32}X_{26}) + \\
&+ Z_{74}(X_{48}Y_{87} - Y_{46}X_{67}) + Z_{85}(X_{59}Y_{98} - Y_{54}X_{48}) + \\
&+ Z_{96}(X_{67}Y_{79} - Y_{65}X_{59})\Big] .
\end{aligned} \tag{2.2}$$

The moduli space of classical vacua of this theory will be given by the critical points of W in the symplectic quotient of field space by the $\mathrm{SU}(N)^9$ action. This space describes deformations of the theory by expectation values for all worldvolume scalars, including both open string states (describing motion of the branes in the transverse space) and twisted closed string states (describing deformations of the orbifold background). These can be separated by the following useful device [5]. The baryonic $\mathrm{U}(1)^8$ global symmetry acts on the space of classical vacua, preserving the symplectic structure. The symplectic quotient of the moduli space by this symmetry, determined by a moment map, yields a space corresponding to the theory with gauge group $\mathrm{U}(N)^9/\mathrm{U}(1)$. This "mesonic" moduli space describes motion of the branes in a deformation of the orbifold determined by the moment map, which here appears as a Fayet-Iliopoulos term. The full moduli space thus fibers over the space $\mathbb{R}^8$ of moment map values, with the fiber over each point a $\mathrm{U}(1)^8$ bundle over the corresponding mesonic moduli space. The degrees of freedom parameterizing this extra structure are the "baryonic" states charged under $\mathrm{U}(1)^8$, and describe the deformations of the orbifold background by twisted closed string states. We will thus find it useful to speak about the theory in terms of Fayet-Iliopoulos terms, keeping in mind that we are in fact describing baryonic moduli.

In the orbifold gauge theory, a non-zero Fayet-Iliopoulos term implies that, in order to satisfy the corresponding D-term equation and preserve supersymmetry, some chiral superfields charged under the corresponding $\mathrm{SU}(N)$ factor must have a non-zero vacuum expectation-value. These vevs generically break some or all of the $\mathrm{SU}(N)^9$ gauge group and introduce a scale into the (previously conformal) orbifold theory. Because of the cubic superpotential (2.2), these vevs also generically induce mass terms for some of the chiral matter. So the resulting Higgsed theory is not conformal, but one assumes that upon renormalization-group flow to the infrared one reaches a conformal point describing the new geometric phase. If the string theory is weakly coupled, the symmetry breaking and integrating out of massive fields can reliably be performed perturbatively.

The work of [9] implies that the four gauge theories describing the cone C over dP_3 can be realized by Fayet-Iliopoulos deformations which have the effect of turning on large, generic vevs for the following four sets of chiral fields in the orbifold theory:

$$\begin{aligned}
(\mathrm{I}):&\quad X_{83}, Y_{79}, Z_{41}\,,\\
(\mathrm{II}):&\quad X_{67}, Y_{87}, Z_{53}\,,\\
(\mathrm{III}):&\quad X_{15}, Y_{32}, Z_{17}\,,\\
(\mathrm{IV}):&\quad X_{26}, Y_{21}, Z_{28}\,.
\end{aligned}\tag{2.3}$$

The fact that each set of vevs involves an X, Y, and Z chiral field follows from the fact that geometrically a $\mathbb{Z}_3$ subgroup of the symmetry group of the orbifold is realized as a symmetry of $C(H)$. However, it is interesting to note that, in terms of the orbifold gauge theory, the sets of vevs in (2.3) break differing amounts of the global symmetry. As we shall see, the effective gauge theories we obtain will be invariant under different global symmetry groups.

Given the expectation values (2.3), we can classically integrate out the massive fields to obtain the effective descriptions at energies below the Higgs scale. The expectation is that all of these will flow in the extreme infrared to the same conformal field theory. Each model has $\mathcal{N}=1$ supersymmetry, a gauge group $\mathrm{SU}(N)^6$, a number of chiral superfields transforming as bi-fundamentals under the gauge group, canonical Kähler terms, and a superpotential.

The representation contents of the four theories (to which we refer as Model I, Model II, Model III, and Model IV) are summarized by the quiver diagrams in figure 1. For each of the four models, we also list below the classical superpotentials (which arise from the

JHEP12(2001)001

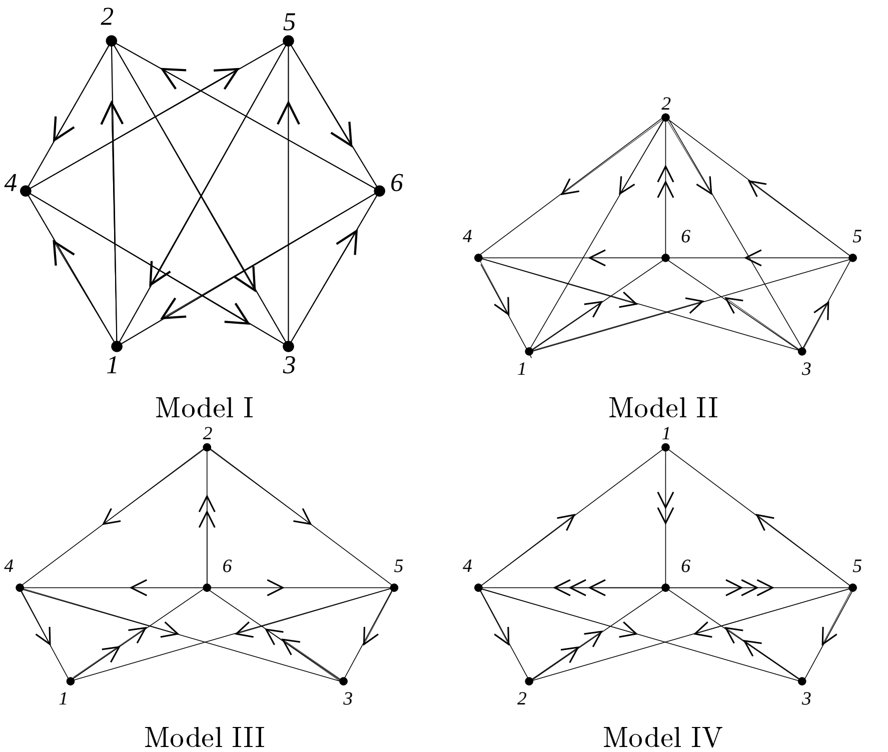

Figure 1: Quivers of the four models.

orbifold superpotential (2.2) after integrating-out massive fields):

$$
\begin{aligned}
W_{\mathrm{I}} &= h_1 \operatorname{tr}\left[X_{32}X_{26}X_{63}\right] + h_2 \operatorname{tr}\left[X_{41}X_{15}X_{54}\right] + h_3 \operatorname{tr}\left[X_{34}X_{42}X_{21}X_{16}X_{65}X_{53}\right] - \\
&\quad - h_4 \operatorname{tr}\left[X_{41}X_{16}X_{63}X_{34}\right] - h_5 \operatorname{tr}\left[X_{21}X_{15}X_{53}X_{32}\right] - h_6 \operatorname{tr}\left[X_{54}X_{42}X_{26}X_{65}\right], \qquad (2.4)\\
W_{\mathrm{II}} &= h_1 \operatorname{tr}\left[Y_{46}Y_{61}X_{14}\right] + h_2 \operatorname{tr}\left[X_{25}X_{51}X_{12}\right] - h_3 \operatorname{tr}\left[Y_{46}X_{63}X_{34}\right] - \\
&\quad - h_4 \operatorname{tr}\left[X_{25}X_{53}X_{32}\right] + h_5 \operatorname{tr}\left[X_{32}X_{26}X_{63}\right] - h_6 \operatorname{tr}\left[X_{12}Y_{26}Y_{61}\right] + \\
&\quad + h_7 \operatorname{tr}\left[X_{34}X_{42}Y_{26}X_{65}X_{53}\right] - h_8 \operatorname{tr}\left[X_{14}X_{42}X_{26}X_{65}X_{51}\right], \qquad (2.5)\\
W_{\mathrm{III}} &= h_1 \operatorname{tr}\left[X_{56}X_{61}X_{15}\right] - h_2 \operatorname{tr}\left[X_{56}Y_{63}X_{35}\right] + h_3 \operatorname{tr}\left[Y_{46}Y_{61}X_{14}\right] - \\
&\quad - h_4 \operatorname{tr}\left[Y_{46}X_{63}X_{34}\right] + h_5 \operatorname{tr}\left[X_{42}Y_{26}Y_{63}X_{34}\right] - h_6 \operatorname{tr}\left[X_{52}Y_{26}Y_{61}X_{15}\right] + \\
&\quad + h_7 \operatorname{tr}\left[X_{52}X_{26}X_{63}X_{35}\right] - h_8 \operatorname{tr}\left[X_{42}X_{26}X_{61}X_{14}\right], \qquad (2.6)\\
W_{\mathrm{IV}} &= h_1 \operatorname{tr}\left[Y_{56}X_{63}X_{35}\right] - h_2 \operatorname{tr}\left[Y_{56}Y_{62}X_{25}\right] + h_3 \operatorname{tr}\left[Y_{46}Y_{61}X_{14}\right] - \\
&\quad - h_4 \operatorname{tr}\left[Y_{46}X_{63}X_{34}\right] + h_5 \operatorname{tr}\left[X_{56}X_{61}X_{15}\right] - h_6 \operatorname{tr}\left[X_{56}Y_{63}X_{35}\right] + \\
&\quad + h_7 \operatorname{tr}\left[X_{46}Y_{62}X_{24}\right] - h_8 \operatorname{tr}\left[X_{46}X_{61}X_{14}\right] + h_9 \operatorname{tr}\left[Z_{56}X_{62}X_{25}\right] - \\
&\quad - h_{10} \operatorname{tr}\left[Z_{56}Y_{61}X_{15}\right] + h_{11} \operatorname{tr}\left[Z_{46}Y_{63}X_{34}\right] - h_{12} \operatorname{tr}\left[Z_{46}X_{62}X_{24}\right]. \qquad (2.7)
\end{aligned}
$$

The parameters h_i are (in general dimensionful) couplings, determined classically in terms of the (generically chosen) expectation values (2.3). The $\mathcal{N} = 1$ supersymmetry ensures that the classical superpotentials will not be renormalized in perturbation theory except via wavefunction renormalization. Varying h_i will correspond to marginal or irrelevant deformations of the theory at weak coupling; we assume this holds in general and thus that a choice of these comprises at most a choice of a point on the critical manifold. We will choose the (nonzero) coefficients to maximize the unbroken classical global symmetry of the model.

We now proceed to study these models and present the evidence relating them to one another as well as degree to which we can demonstrate that they reproduce the properties expected from the geometry of X and H (to be discussed in the next section).

2.1 Chiral operators

We first compare the algebras of chiral operators in the four models. In any quiver theory containing only bi-fundamental chiral fields, the gauge-invariant operators fall into two classes. First, for each chiral operator $\mathcal{O}$ which transforms under the various factors of the gauge-group as an adjoint (or a singlet), we have the gauge-invariant operator $\mathrm{tr}[\mathcal{O}]$. We refer to these operators as mesonic operators. Second, since only $\mathrm{SU}(N)$ factors occur in the gauge-group (the $\mathrm{U}(1)$ factors of a naive $\mathrm{U}(N)$ are broken as discussed above), we may use the invariant anti-symmetric tensor to make baryonic gauge-invariant chiral operators. If $\mathcal{O}$ is a chiral operator transforming as a fundamental or anti-fundamental (or singlet) under each $\mathrm{SU}(N)$ factor, then we form a gauge-invariant baryonic operator by anti-symmetrizing on each gauge-index, i.e. in the case of a bi-fundamental operator, $\epsilon_{i_1 \ldots i_N} \epsilon^{j_1 \cdots j_N} (\mathcal{O})^{i_1}_{j_1} \ldots (\mathcal{O})^{i_N}_{j_N}$. We abbreviate such an operator by $\mathcal{O}^N$. In general, we may have multiple chiral operators $\mathcal{O}_1$, $\mathcal{O}_2$, etc. transforming in the same representation. For simplicity, assuming $\mathcal{O}_1$ and $\mathcal{O}_2$ are both bi-fundamentals, then we may also make baryonic operators of the form $\epsilon_{i_1 \ldots i_N} \epsilon^{j_1 \cdots j_N} (\mathcal{O}_1)^{i_1}_{j_1} \ldots (\mathcal{O}_1)^{i_s}_{j_s} (\mathcal{O}_2)^{i_{s+1}}_{j_{2+1}} \ldots (\mathcal{O}_2)^{i_N}_{j_N}$. We abbreviate such operators by $\mathcal{O}_1^s \mathcal{O}_2^{N-s}$, and similarly for baryonic operators involving more than two flavors.

As discussed above, the baryonic operators are associated to deformations of the closed string background by twisted fields. The mesonic operators, on the other hand, will describe motion of the branes. In terms of the holographically dual AdS dual, the baryonic operators, which carry $\mathcal{R}$-charge (and hence conformal weight) proportional to N, are associated to stringy effects, and the baryons themselves correspond to wrapped branes. Hence we first consider the mesonic operators. The chiral ring is (classically) isomorphic to the ring of gauge-invariant holomorphic functions in the fields modulo the jacobian ideal of W. In each of the theories the mesonic ring is generated by seven operators $\mathcal{M}_i$. In the appendices, we list these as functions of the fields for each model.

A first check of the claimed duality is that the subalgebras of the chiral rings in each of the four models which are generated by the mesonic invariants agree (and do realize the geometry of the cone C). It is easy, if somewhat tedious, to check that in each of the four models all relations between the mesonic generators $\mathcal{M}_i$ are generated by the following nine quadratic polynomials:

$$\begin{array}{lll}\mathcal{M}_1\mathcal{M}_7 - \mathcal{M}_2\mathcal{M}_6\,, & \mathcal{M}_2\mathcal{M}_7 - \mathcal{M}_1\mathcal{M}_3\,, & \mathcal{M}_3\mathcal{M}_7 - \mathcal{M}_2\mathcal{M}_4\,, \\ \mathcal{M}_4\mathcal{M}_7 - \mathcal{M}_3\mathcal{M}_5\,, & \mathcal{M}_5\mathcal{M}_7 - \mathcal{M}_4\mathcal{M}_6\,, & \mathcal{M}_6\mathcal{M}_7 - \mathcal{M}_1\mathcal{M}_5\,, \\ \mathcal{M}_7^2 - \mathcal{M}_1\mathcal{M}_4\,, & \mathcal{M}_7^2 - \mathcal{M}_2\mathcal{M}_5\,, & \mathcal{M}_7^2 - \mathcal{M}_3\mathcal{M}_6\,. \end{array} \tag{2.8}$$

As we will show in section 3, the ideal generated by these nine polynomials also presents the cone C as a subvariety of $\mathbb{C}^7$ (identifying the coordinates of $\mathbb{C}^7$ with the mesonic generators $\mathcal{M}_i$ in the above). So the invariant traces of products of the mesonic generators parametrize the S_N invariants of N points satisfing (2.8), and we identify the mesonic branch of the moduli space as the symmetric product of N copies of the cone C (as we expect for N identical branes placed transverse to C). Our investigation of the mesonic subalgebra of the chiral ring is thus consistent with the conjecture that the models are "dual" and flow to the same infrared fixed point.

2.2 Global symmetries

We now examine the global symmetries in each of the four models. The continuous global symmetries are abelian. Each of the models has a nonanomalous $\mathrm{U}(1)^6$ global symmetry, and the charges of the chiral superfields under this are listed in table 1. In this table we have chosen a basis for this $\mathrm{U}(1)^6$ in each model, and we have labelled the individual factors as $\mathrm{U}(1)_A$, $\mathrm{U}(1)_B$, $\mathrm{U}(1)_C$, $\mathrm{U}(1)_D$, $\mathrm{U}(1)_E$, and $\mathrm{U}(1)_\mathcal{R}$. The last of these, of course, acts as an R-symmetry. The identification of these bases between models is nontrivial. The mesonic operators in all four theories are charged only under the subgroup $\mathrm{U}(1)_A \times \mathrm{U}(1)_B \times \mathrm{U}(1)_\mathcal{R}$, with charges

$$\left(\begin{array}{ccccccc} \mathcal{M}_1 & \mathcal{M}_2 & \mathcal{M}_3 & \mathcal{M}_4 & \mathcal{M}_5 & \mathcal{M}_6 & \mathcal{M}_7 \\ 0 & -1 & -1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 & -1 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{array}\right). \tag{2.9}$$

We will see that the action of this $\mathrm{U}(1)^3$ on the moduli space in the gauge theory is in agreement with an expected $\mathrm{U}(1)^3$ isometry of the horizon. Under the additional $\mathrm{U}(1)_C \times \mathrm{U}(1)_D \times \mathrm{U}(1)_E$ symmetries, only the baryonic operators in each model are charged. Such a baryonic symmetry corresponds in the dual AdS theory to $\mathrm{U}(1)$ gauge symmetries arising from the reduction of the R-R four-form about three-cycles on H. We will discuss these cycles in section 3 and will find agreement with this prediction. In section 4, we will return to this symmetry as we identify baryons with wrapped D3-branes.

Although we will examine the baryonic chiral operators in more detail in section 4, we make a few remarks now. The chiral algebra of baryonic operators carries a $\mathbb{Z}$-grading by $\mathcal{R}$-charge, or equivalently conformal-weight, and we will restrict our discussion to the three lowest levels of baryonic operators corresponding to $\mathcal{R}$-charge $\frac{1}{3}N$, $\frac{2}{3}N$, and N. In the appendices we list generators $\mathcal{B}_i$, $\mathcal{B}_i(s)$, and $\mathcal{B}_i(s,t)$ of the classical rings of baryonic operators for these three levels. Note that in each of the four models we find the same number of generators at each level, with corresponding generators having the same charges under the global $\mathrm{U}(1)^6$.

Model I

	A	B	C	D	E	R
X_{21}	1	1	1	0	1	1/3
X_{65}	0	0	−1	−1	1	1/3
X_{34}	0	0	0	1	1	1/3
X_{53}	0	1	1	0	−1	1/3
X_{42}	0	−1	−1	−1	−1	1/3
X_{16}	−1	−1	0	1	−1	1/3
X_{32}	−1	−1	−1	0	0	2/3
X_{15}	0	−1	−1	0	0	2/3
X_{26}	0	1	1	1	0	2/3
X_{54}	0	0	1	1	0	2/3
X_{63}	1	0	0	−1	0	2/3
X_{41}	0	1	0	−1	0	2/3

Model II

	A	B	C	D	E	R
X_{14}	1	1	1	0	1	1/3
X_{65}	0	0	−1	−1	1	1/3
X_{34}	0	0	0	1	1	1/3
X_{53}	0	1	1	0	−1	1/3
X_{42}	0	−1	−1	−1	−1	1/3
X_{51}	−1	−1	0	1	−1	1/3
X_{32}	−1	−1	−1	0	0	2/3
Y_{61}	0	−1	−1	0	0	2/3
X_{26}	0	1	1	1	0	2/3
Y_{26}	0	0	1	1	0	2/3
X_{63}	1	0	0	−1	0	2/3
X_{12}	0	1	0	−1	0	2/3
Y_{46}	−1	0	0	0	−1	1
X_{25}	1	0	0	0	1	1

Model III

	A	B	C	D	E	R
X_{14}	1	1	1	0	1	1/3
X_{52}	0	0	−1	−1	1	1/3
X_{34}	0	0	0	1	1	1/3
X_{15}	0	1	1	0	−1	1/3
X_{42}	0	−1	−1	−1	−1	1/3
X_{35}	−1	−1	0	1	−1	1/3
X_{61}	−1	−1	−1	0	0	2/3
Y_{61}	0	−1	−1	0	0	2/3
X_{26}	0	1	1	1	0	2/3
Y_{26}	0	0	1	1	0	2/3
X_{63}	1	0	0	−1	0	2/3
Y_{63}	0	1	0	−1	0	2/3
Y_{46}	−1	0	0	0	−1	1
X_{56}	1	0	0	0	1	1

Model IV

	A	B	C	D	E	R
X_{14}	1	1	1	0	1	1/3
X_{24}	0	0	−1	−1	1	1/3
X_{34}	0	0	0	1	1	1/3
X_{15}	0	1	1	0	−1	1/3
X_{25}	0	−1	−1	−1	−1	1/3
X_{35}	−1	−1	0	1	−1	1/3
X_{61}	−1	−1	−1	0	0	2/3
Y_{61}	0	−1	−1	0	0	2/3
X_{62}	0	1	1	1	0	2/3
Y_{62}	0	0	1	1	0	2/3
X_{63}	1	0	0	−1	0	2/3
Y_{63}	0	1	0	−1	0	2/3
X_{46}	0	0	0	0	−1	1
Y_{46}	−1	0	0	0	−1	1
Z_{46}	0	−1	0	0	−1	1
X_{56}	1	0	0	0	1	1
Y_{56}	0	1	0	0	1	1
Z_{56}	0	0	0	0	1	1

Table 1: $U(1)^6$ charges in Models I, II, III, and IV.

We now discuss the maximal discrete global symmetries of the four models (corresponding to the most symmetric choice of the couplings h_i and gauge-couplings in each model). As we will explain in section 3, the order-12 dihedral group[4] D_{12} acts as a sym-

[4]This group acts as the symmetries of a regular hexagon and is isomorphic to $S_3 \times \mathbb{Z}_2$.

JHEP12(2001)001

metry of X, and it also acts as a symmetry of the non-spherical horizon H and the cone C. So we expect to see a D_{12} discrete symmetry in the four models. In fact, we only see this symmetry in two of the models.

In each model, the discrete symmetries act by permutations of the $\mathrm{SU}(N)$ factors in the gauge group, along with appropriate permutations of the chiral superfields and possibly charge-conjugation. So we specify these symmetries by supplying a cycle-decomposition of these permutations between $\mathrm{SU}(N)$ factors (and also the corresponding action on chiral fields where this is not uniquely determined by the charges), and we denote charge-conjugation by $\mathcal{C}$. For instance, in Model I we see that the expected D_{12} symmetry is generated by the following order-6 and order-2 elements:

$$(612435)\,, \qquad \text{and} \qquad (45)(26)\mathcal{C}\,. \tag{2.10}$$

On the mesonic generators $\mathcal{M}_i$ these act as the permutations

$$(123456)\,, \qquad \text{and} \qquad (26)(35)\,. \tag{2.11}$$

As we shall see, this agrees precisely with the D_{12} action on X and H.

In the other three models, these discrete symmetries also include an action by phases on the chiral fields. These additional phases arise under those permutations which are $\mathcal{R}$-symmetries, for which the superpotential transforms as $W \mapsto -W$. In order to preserve the sign of W, we must include with these permutations a factor $\exp(i\frac{\pi}{2}r)$, where $\exp(i\frac{\pi}{2}r)$ is the naive $\mathcal{R}$-symmetry which acts on the gauginos as $\lambda \mapsto i\lambda$, on all chiral fermions as $\psi \mapsto -i\psi$, and fixes the scalar components of the chiral fields. Generally, though, the combined permutation and rotation by the naive $\mathcal{R}$-symmetry will be anomalous (since the naive $\mathcal{R}$-symmetry itself is anomalous in these models). A similar situation was encountered in [16], but in that case the outer-automorphism of the gauge-group was enough to ensure that the discrete symmetry was non-anomalous. In these models, in order to find a non-anomalous discrete symmetry we must explicitly cancel the anomaly in the naive discrete symmetry with another anomaly arising from an additional $\mathrm{U}(1)_a$ action. This $\mathrm{U}(1)_a$ action manifests itself as the additional phases on chiral fields to which we alluded above.

For instance, in Model II we have the following discrete symmetries:

$$(45)(26)\mathcal{C}\,, \qquad (13)(X_{26}Y_{26})\exp\left(i\frac{\pi}{2}r\right)\exp(i\pi a)\,. \tag{2.12}$$

First, note that we do not see the full D_{12} as expected, but only a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup. Second, the symmetry involving the permutation (13) is an $\mathcal{R}$-symmetry, for which we have included the factor $\exp(i\frac{\pi}{2}r)$. The naive $\mathcal{R}$-symmetry is anomalous, which means that in the presence of k_j instantons in the j-th $\mathrm{SU}(N)$ factor of the gauge group, the path-integral measure transforms with a factor of i for each of the $2Nk_j$ gaugino zero-modes associated to $\mathrm{SU}(N)_j$ and a factor of $-i$ for each of the Nk_j zero-modes of each chiral field charged under $\mathrm{SU}(N)_j$. This anomaly implies that the measure picks up an overall phase $(-1)^{N(k_1+k_3+k_4+k_5)}$ under $\exp(i\frac{\pi}{2}r)$. Since the outer-automorphism (13) of $\mathrm{SU}(N)^6$ exchanges the first and third $\mathrm{SU}(N)$ factors, the classical instanton background which contributes to the anomaly satisfies $k_1 = k_3$ — but we still have a possible anomalous phase arising from instantons in the fourth and fifth $\mathrm{SU}(N)$ factors. We cancel this anomaly with the (anomalous) $\mathbb{Z}_2$ symmetry $\exp(i\pi a)$ under which the fields X_{14}, X_{34}, X_{51}, Y_{26}, X_{12}, and Y_{46} transform by a phase -1. Under this $\mathbb{Z}_2$, the zero-modes of each chiral field contribute a phase of $(-1)^{N(k_1+k_3+k_4+k_5)}$ to cancel the anomaly. On the mesonic operators $\mathcal{M}_i$, this generator then acts as

$$\begin{aligned}
&\mathcal{M}_1 \to -\mathcal{M}_4\,, \quad && \mathcal{M}_4 \to -\mathcal{M}_1\,, && \\
&\mathcal{M}_2 \to \mathcal{M}_3\,, && \mathcal{M}_3 \to -\mathcal{M}_2\,, && \\
&\mathcal{M}_5 \to \mathcal{M}_6\,, && \mathcal{M}_6 \to -\mathcal{M}_5\,, \qquad && \mathcal{M}_7 \to \mathcal{M}_7\,,
\end{aligned} \tag{2.13}$$

corresponding to the action of the order-2 element of S_3 in (2.11) and a π-rotation in the second U(1) subgroup of (2.9) (hence we could remove the phases above by including with the $\mathbb{Z}_2$ an action of this U(1)).

In Model III, we also see only a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup of D_{12}. The generators for this symmetry are

$$(45)(X_{61}Y_{61})(X_{63}Y_{63})(X_{26}Y_{26})\,, \qquad (13)(X_{26}Y_{26})\exp\left(i\frac{\pi}{2}r\right)\exp(i\pi a)\,. \tag{2.14}$$

Here $\exp(i\pi a)$ acts with a phase of (-1) on the fields X_{34}, X_{61}, X_{63}, Y_{63}, X_{26}, and X_{56}. The action on the mesonic operators $\mathcal{M}_i$ is as in (2.13).

Finally, in Model IV we again see the full D_{12} discrete symmetry. The order-6 and order-2 elements generating this symmetry are respectively

$$\begin{aligned}
&(123)(45)(X_{56}X_{46}Y_{56}Y_{46}Z_{56}Z_{46})(X_{61}Y_{62}X_{63}Y_{61}X_{62}Y_{63})\,, \\
&(12)(X_{56}Y_{56})(Y_{46}Z_{46})(X_{61}Y_{62})(Y_{61}X_{62})(X_{63}Y_{63})\exp(i\pi r)\exp(i\pi a)\,.
\end{aligned} \tag{2.15}$$

Here $\exp(i\pi a)$ acts with a phase (-1) on the fields charged under the fourth $\mathrm{SU}(N)$, i.e. X_{14}, X_{24}, X_{34}, X_{46}, Y_{46}, and Z_{46}. The D_{12} action on the $\mathcal{M}_i$ in Model IV is as in (2.11).

Two aspects of the global symmetry of the four models seem to challenge the assertion that they all flow to the same conformal fixed point. The first, that only two of the models exhibit the full expected discrete symmetry, is not very serious. The $\mathbb{Z}_3$ generator absent from Model II and III could be an accidental symmetry which is restored in the infrared.

The second puzzle involves the 't Hooft anomaly matching conditions. One of the standard checks for duality relations among theories flowing to identical conformal fixed points involves the equality of the anomalies in global symmetries. A quick look at table 1, however, shows that while the chiral multiplet charges under the global symmetries agree to a remarkable extent (recall these charged fields do not represent local fields in the conformal field theory) there are, in each of the theories II, III, and IV, additional fields of $\mathcal{R}$-charge one which lead to a failure of anomaly matching (except between models II and III in which the charge spectra are identical). Thus, the anomaly coefficients computed in the perturbative limit do not agree, and the four models fail this standard and basic test of duality.

A closer inspection, however, reveals that the problem is that we are being too simple-minded in our interpretation of the physics described by models II, III, and IV. The perturbative calculation of the anomaly coefficients used above is valid when all the gauge couplings are weak. In model I, in which all of the factors of the gauge group are asymptotically free, we can construct an RG trajectory ending at the fixed point, beginning in the deep UV at weak coupling. In each of the other three models, the one-loop beta functions for the various factors of the gauge groups have different signs. Thus, a trajectory ending at the fixed point and containing a weakly coupled point will in general not exist. Thus, the 't Hooft condition that anomaly coefficients are independent of scale does not permit us to perform perturbative calculations which will hold at the IR fixed point.

A closer look at table 1 reveals that some anomaly coefficients, namely those including at least one factor of $\mathrm{U}(1)_{\mathcal{R}}$, agree between the models. This is because the only discrepancy in the table involves chiral multiplets of $\mathcal{R}$-charge one, hence chiral fermions that are $\mathcal{R}$-neutral. In particular, this includes the coefficient of the $\mathrm{U}(1)^3_{\mathcal{R}}$ anomaly, related to the central charge of the conformal theory and, via the AdS/CFT duality, to the volume of H.

Following the methods of [17], we compare the respective central charges and horizon volumes in the del Pezzo and $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ orbifold theories. We first recall some facts about $\mathcal{N} = 1$ superconformal gauge theories in $d = 4$. In these theories, the trace of the stress-energy tensor receives anomalous contributions

$$\langle T^\mu_\mu \rangle = -aE_4 - cI_4 \,, \tag{2.16}$$

where E_4 and I_4 are terms quadratic in the Riemann tensor, and a and c (the central charge) are anomaly coefficients. If R_μ denotes the $\mathcal{R}$-symmetry current, then the anomaly coefficients a and c are related to the $\mathrm{U}(1)_{\mathcal{R}}$-gravitational and $\mathrm{U}(1)^3_{\mathcal{R}}$ anomalies [18],

$$\begin{aligned} (a-c) \;&\sim\; \langle (\partial_\mu R^\mu) T_{\alpha\beta} T_{\gamma\delta} \rangle \sim \sum_\psi \mathcal{R}(\psi) \,, \\ (5a-3c) \;&\sim\; \langle (\partial_\mu R^\mu) R_\alpha R_\beta \rangle \sim \sum_\psi \Big[\mathcal{R}(\psi)\Big]^3 . \end{aligned} \tag{2.17}$$

If we evaluate these anomaly coefficients perturbatively in each of the four models,[5] summing over the $6N^2$ gluinos of $\mathcal{R}$-charge 1 and two sets of $6N^2$ matter fermions of $\mathcal{R}$-charges $-2/3$ and $-1/3$, we see that $a = c$ and $c \sim 2N^2$.

If we perform the same calculation of the central charge in the $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ orbifold gauge theory, we find $c \sim 4N^2$. As observed by Gubser [19], in such a situation where we deform the orbifold gauge theory in the UV and flow to the IR dP_3 theory, we expect a relation from the AdS/CFT correspondence

$$\frac{c_{IR}}{c_{UV}} = \frac{1/\mathrm{Vol}(H)}{1/\mathrm{Vol}(S^5/(\mathbb{Z}_3 \times \mathbb{Z}_3))} \tag{2.18}$$

between ratios of central charges and respective volumes of horizons. We have just computed $c_{IR} : c_{UV}$ as $1:2$. In section 3, we will compute the ratio of volumes $\mathrm{Vol}(S^5/(\mathbb{Z}_3 \times \mathbb{Z}_3)) : \mathrm{Vol}(H)$, which we find to be $1:2$ as well.

[5] We believe that only in Model I is there a scale at which this perturbative analysis is valid, and it seems to be a coincidence that the other models agree.

2.3 $\mathcal{N}=1$ duality in the four models

In this section, we show that in fact the four models are related by the $\mathcal{N}=1$ duality of [11]. This duality, of course, relates SQCD-like theories flowing to the same conformal fixed point in the deep IR. Moreover, it provides a mapping of chiral operators between the two models that can be used to map deformations of one into "dual" deformations of the other, preserving the property that the physics at extremely low energies is identical. What we mean by $\mathcal{N}=1$ duality in the context of our models is the following. Consider, in any of the models, a factor of the gauge group under which a total of $2N$ chiral multiplets transform in the fundamental representation. If we deform the theory by turning off the superpotential couplings for these fields, as well as the gauge coupling for any other factors in the gauge group under which they are charged, we isolate a SQCD-like theory. In this deformed theory, [11] predicts an equivalent description in terms of "magnetic" variables transforming under a magnetic gauge group $\mathrm{SU}(N)$ (hence the insistence on $2N$ fundamentals, since we wish to remain with quivers of this type). One can then turn on the remaining couplings, rewritten in terms of the magnetic variables, returning to a dual description of the original theory. One of the most striking properties of the $\mathcal{N}=1$ duality is the fact that such mappings of deformations lead to agreement between two dual models. We will see that this is indeed the case.

So in order to apply the $\mathcal{N}=1$ duality to our models with a product gauge-group and superpotential, we start in Model I by going to a point[6] corresponding to a pure $\mathrm{SU}(N)$ SQCD theory, for which all factors of the gauge-group but one, suppose the first, decouple and the superpotential is turned off. In this pure $\mathrm{SU}(N)$ theory we apply the usual $\mathcal{N}=1$ transformation to obtain a dual description.

Thus we define composite mesonic fields (μ an arbitrary mass scale)

$$\begin{aligned}
\mu M_{25} &= X_{21} X_{15}\,, \\
\mu M_{26} &= X_{21} X_{16}\,, \\
\mu M_{45} &= X_{41} X_{15}\,, \\
\mu M_{46} &= X_{41} X_{16}\,,
\end{aligned} \tag{2.19}$$

and introduce the dual chiral fields $\tilde{X}_{12}$, $\tilde{X}_{14}$, $\tilde{X}_{51}$, and $\tilde{X}_{61}$. The dual $\mathrm{SU}(N)$ SQCD model possesses a cubic superpotential

$$\lambda\Big\{ \mathrm{tr}\left[M_{46}\tilde{X}_{61}\tilde{X}_{14}\right] - \mathrm{tr}\left[M_{45}\tilde{X}_{51}\tilde{X}_{14}\right] + \mathrm{tr}\left[M_{25}\tilde{X}_{51}\tilde{X}_{12}\right] - \mathrm{tr}\left[M_{26}\tilde{X}_{61}\tilde{X}_{12}\right]\Big\}\,. \tag{2.20}$$

The relative signs in (2.20) are fixed by the presence in the original SQCD theory of a global $\mathrm{SU}(2)\times\mathrm{SU}(2)$ symmetry under which X_{51} and X_{61} (resp. X_{12} and X_{14}) transform as doublets.

[6] Note that this point can only be reached via relevant deformations of the theory, and hence lies away from the critical manifold of interest. However, the dual SQCD theories will possess dual relevant deformations which may be used to reach the critical manifold.

JHEP12(2001)001

We now restore the original superpotential and the remaining gauge couplings. In the dual model this leads to the superpotential

$$\begin{aligned}\widetilde{W}_{\mathrm{I}} &= h_1 \operatorname{tr}\Big[X_{32}X_{26}X_{63}\Big] + h_2\mu \operatorname{tr}\Big[M_{45}X_{54}\Big] + h_3\mu \operatorname{tr}\Big[X_{34}X_{42}M_{26}X_{65}X_{53}\Big] - \\ &\quad -h_4\mu \operatorname{tr}\Big[M_{46}X_{63}X_{34}\Big] - h_5\mu \operatorname{tr}\Big[M_{25}X_{53}X_{32}\Big] - h_6 \operatorname{tr}\Big[X_{54}X_{42}X_{26}X_{65}\Big] + \\ &\quad +\lambda\Big\{ \operatorname{tr}\Big[M_{46}\tilde{X}_{61}\tilde{X}_{14}\Big] - \operatorname{tr}\Big[M_{45}\tilde{X}_{51}\tilde{X}_{14}\Big] + \operatorname{tr}\Big[M_{25}\tilde{X}_{51}\tilde{X}_{12}\Big] - \operatorname{tr}\Big[M_{26}\tilde{X}_{61}\tilde{X}_{12}\Big]\Big\}.\end{aligned} \tag{2.21}$$

We now integrate-out the massive fields M_{45} and X_{54} from (2.21) to obtain an equivalent effective potential for the low-energy dual theory

$$\begin{aligned}\widetilde{W}_{\mathrm{I}}^{\mathrm{eff}} &= h_1 \operatorname{tr}\Big[X_{32}X_{26}X_{63}\Big] + h_3\mu \operatorname{tr}\Big[X_{34}X_{42}M_{26}X_{65}X_{53}\Big] - \\ &\quad -h_4\mu \operatorname{tr}\Big[M_{46}X_{63}X_{34}\Big] - h_5\mu \operatorname{tr}\Big[M_{25}X_{53}X_{32}\Big] - 2\frac{h_6\lambda}{h_2\mu} \operatorname{tr}\Big[\tilde{X}_{51}\tilde{X}_{14}X_{42}X_{26}X_{65}\Big] + \\ &\quad +\lambda \operatorname{tr}\Big[M_{46}\tilde{X}_{61}\tilde{X}_{14}\Big] + \lambda \operatorname{tr}\Big[M_{25}\tilde{X}_{51}\tilde{X}_{12}\Big] - \lambda \operatorname{tr}\Big[M_{26}\tilde{X}_{61}\tilde{X}_{12}\Big].\end{aligned} \tag{2.22}$$

We observe that $\widetilde{W}_{\mathrm{I}}^{\mathrm{eff}}$ is the superpotential for Model II and that the representation contents of the models agree. Thus deforming back to the original model, we have found that Model I is dual to Model II. We also observe that the fundamental chiral superfields Y_{26}, Y_{46}, and X_{25} in Model II arise under the duality from the mesonic fields M_{26}, M_{46}, and M_{25} in Model I. Under the inverse duality transformation, we of course obtain Model I again. In this case we find that the fundamental field X_{54} in Model I arises from a mesonic composite field in Model II.

We now consider the relation between Model II and Model III, with similar reasoning. In this case, we consider the SQCD point in the deformation space of Model II for which all SU(N) factors decouple but the one labelled '5' and for which the superpotential is turned off. At this point, we again define the composite mesonic fields

$$\begin{aligned}\mu\, M_{21} &= X_{25}X_{51}\,, \\ \mu\, M_{23} &= X_{25}X_{53}\,, \\ \mu\, M_{61} &= X_{65}X_{51}\,, \\ \mu\, M_{63} &= X_{65}X_{53}\,,\end{aligned} \tag{2.23}$$

and the dual fields $\tilde{X}_{15}$, $\tilde{X}_{35}$, $\tilde{X}_{52}$, and $\tilde{X}_{56}$. The dual superpotential $\widetilde{W}_{\mathrm{II}}$ then is

$$\begin{aligned}\widetilde{W}_{\mathrm{II}} &= h_1 \operatorname{tr}\Big[Y_{46}Y_{61}X_{14}\Big] + h_2\mu \operatorname{tr}\Big[M_{21}X_{12}\Big] - h_3 \operatorname{tr}\Big[Y_{46}X_{63}X_{34}\Big] - \\ &\quad - h_4\mu \operatorname{tr}\Big[M_{23}X_{32}\Big] + h_5 \operatorname{tr}\Big[X_{32}X_{26}X_{63}\Big] - h_6 \operatorname{tr}\Big[X_{12}Y_{26}Y_{61}\Big] + \\ &\quad + h_7\mu \operatorname{tr}\Big[X_{34}X_{42}Y_{26}M_{63}\Big] - h_8\mu \operatorname{tr}\Big[X_{14}X_{42}X_{26}M_{61}\Big] + \\ &\quad + \lambda\Big\{ \operatorname{tr}\Big[M_{23}\tilde{X}_{35}\tilde{X}_{52}\Big] - \operatorname{tr}\Big[M_{63}X_{35}X_{56}\Big] + \operatorname{tr}\Big[M_{61}X_{15}X_{56}\Big] - \operatorname{tr}\Big[M_{21}\tilde{X}_{15}\tilde{X}_{52}\Big]\Big\}.\end{aligned} \tag{2.24}$$

Integrating-out the massive fields M_{21}, X_{12}, M_{23}, and X_{32} yields a low-energy effective superpotential

$$\begin{aligned}\widetilde{W}_{\rm II}^{\rm eff} = & \, h_1 \,\mathrm{tr}\left[Y_{46}Y_{61}X_{14}\right] - h_3 \,\mathrm{tr}\left[Y_{46}X_{63}X_{34}\right] + \lambda \,\mathrm{tr}\left[M_{61}X_{15}X_{56}\right] - \lambda \,\mathrm{tr}\left[M_{63}X_{35}X_{56}\right] + \\ & + h_7\mu \,\mathrm{tr}\left[X_{34}X_{42}Y_{26}M_{63}\right] - h_8\mu \,\mathrm{tr}\left[X_{14}X_{42}X_{26}M_{61}\right] + \\ & + 2\frac{h_5\lambda}{h_4\mu} \,\mathrm{tr}\left[\tilde{X}_{35}\tilde{X}_{52}X_{26}X_{63}\right] - 2\frac{h_6\lambda}{h_2\mu} \,\mathrm{tr}\left[\tilde{X}_{15}\tilde{X}_{52}Y_{26}Y_{61}\right]. \end{aligned} \tag{2.25}$$

We again see that this effective superpotential is the superpotential of Model III and the representation contents agree. Further, we note that the fundamental superfields X_{61} and Y_{63} of Model III arise under the duality from the composite fields M_{61} and M_{63} of Model II. Under the inverse duality transformation, the fields X_{32} and X_{12} in Model II arise as composite mesons in Model III.

Finally, to relate Model III to Model IV, we decouple all the SU(N) factors of the gauge group in Model II except factor '2' and turn off the superpotential. At this point we define the composite mesonic fields

$$\begin{aligned}\mu\, M_{46} &= X_{42}X_{26}\,, \\ \mu\, M_{56} &= X_{52}X_{26}\,, \\ \mu\, N_{46} &= X_{42}Y_{26}\,, \\ \mu\, N_{56} &= X_{52}Y_{26}\,,\end{aligned} \tag{2.26}$$

and introduce the dual fields $\tilde{X}_{24}$, $\tilde{X}_{25}$, $\tilde{X}_{62}$, and $\tilde{Y}_{62}$. The corresponding dual superpotential is

$$\begin{aligned}\widetilde{W}_{\rm III} = & \, h_1 \,\mathrm{tr}\left[X_{56}X_{61}X_{15}\right] - h_2 \,\mathrm{tr}\left[X_{56}Y_{63}X_{35}\right] + h_3 \,\mathrm{tr}\left[Y_{46}Y_{61}X_{14}\right] - \\ & - h_4 \,\mathrm{tr}\left[Y_{46}X_{63}X_{34}\right] + h_5\mu \,\mathrm{tr}\left[N_{46}Y_{63}X_{34}\right] - h_6\mu \,\mathrm{tr}\left[N_{56}Y_{61}X_{15}\right] + \\ & + h_7\mu \,\mathrm{tr}\left[M_{56}X_{63}X_{35}\right] - h_8\mu \,\mathrm{tr}\left[M_{46}X_{61}X_{14}\right] + \\ & + \lambda\Big\{ \mathrm{tr}\left[M_{46}\tilde{Y}_{62}\tilde{X}_{24}\right] - \mathrm{tr}\left[M_{56}\tilde{Y}_{62}\tilde{X}_{25}\right] + \mathrm{tr}\left[N_{56}\tilde{X}_{62}\tilde{X}_{25}\right] - \mathrm{tr}\left[N_{46}\tilde{X}_{62}\tilde{X}_{24}\right]\Big\}. \end{aligned} \tag{2.27}$$

We identify $\widetilde{W}_{\rm III}$ with the superpotential of Model IV and note that the representations also agree. We also note that in this case the fundamental fields Y_{56}, Z_{56}, X_{46}, and Z_{46} of Model IV have appeared from the composite fields M_{56}, N_{56}, M_{46}, and N_{46}. Under the inverse duality transformation, we reproduce Model III. None of the fundamental fields in Model III arise as composites in Model IV.

So, by going to various points at which we can apply the $\mathcal{N}=1$ duality, we see that the four models do describe the same family of conformal theories. In [15] the authors mention two other pairs of "toric dual" theories, describing D3-branes transverse to a cone over a del Pezzo surface dP_2 which is the blowup of $\mathbb{P}^2$ at two points and to a cone over a Hirzebruch surface $\mathbb{F}_0$ (better known as $\mathbb{P}^1 \times \mathbb{P}^1$). We will indicate how these theories are related via $\mathcal{N}=1$ duality in section 5. We are led to conjecture that all instances of "toric duality" are in fact of this form.

JHEP12(2001)001

3. The geometry and topology of X and H

In this section, we must discuss some rather detailed geometry of $X = dP_3$ and the associated non-spherical horizon H. This is necessary to understand the symmetries and especially the baryonic operator content of the four models. However, the result is that we will able to do a very good job (in section 4) of understanding the quantum ring of baryonic operators.

We begin with the complex surface X, which can be described as the blowup of $\mathbb{P}^2$ at three generic (i.e. non-colinear) points. X is a smooth projective surface and a toric variety, and we have a number of ways to understand its geometry. We will proceed classically, using a few facts about the geometry of non-singular projective surfaces [20, 21]. (For a toric description, see for instance [9], or more generally [22].)

3.1 The geometry and topology of X

We start with $\mathbb{P}^2$ having homogeneous coordinates $[x : y : z]$. We blow up $\mathbb{P}^2$ at three points, which without loss we may assume to be $P_1 = [1 : 0 : 0]$, $P_2 = [0 : 1 : 0]$, and $P_3 = [0 : 0 : 1]$. The resulting surface is X, which we can present via the blow-up as a complex subvariety of $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Let $[u_1 : v_1]$, $[u_2 : v_2]$, and $[u_3 : v_3]$ be homogeneous coordinates on the respective $\mathbb{P}^1$ factors above. Then X is the subvariety determined by the following homogeneous equations

$$y v_1 - z u_1 = 0\,, \qquad z v_2 - x u_2 = 0\,, \qquad x v_3 - y u_3 = 0\,. \tag{3.1}$$

Note that away from the points P_1, P_2, and P_3, the three equations in (3.1) determine a unique point in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and so X is locally isomorphic to $\mathbb{P}^2$ on this open set. Conversely, for any one of those points P_1, P_2, or P_3 on $\mathbb{P}^2$, one of the equations in (3.1) is identically satisfied for all points on one $\mathbb{P}^1$ factor. Thus on dP_3, the points P_1, P_2, and P_3 of $\mathbb{P}^2$ are replaced by three "exceptional" $\mathbb{P}^1$'s. We also have a natural surjection, the blow-down, $\pi : X \to \mathbb{P}^2$, which maps each of the three $\mathbb{P}^1$'s to the points P_1, P_2, and P_3 and is the natural identification everywhere else.

We are interested really in the complex cone C over X. We can see this cone explicitly if we embed X as a subvariety of some projective space $\mathbb{P}^N$. Then the homogeneous equations specifying the image of the embedding $X \hookrightarrow \mathbb{P}^N$ can also be understood as equations for the embedding of the cone C in $\mathbb{C}^{N+1}$.

To find a good embedding $C \hookrightarrow \mathbb{P}^N$, we consider the linear system of cubic curves in $\mathbb{P}^2$ which pass through the three points P_1, P_2, and P_3. This system is associated to a linear subspace E of the vector space $\Gamma(\mathbb{P}^2, \mathcal{O}(3))$ of global sections of $\mathcal{O}(3)$. We can easily see that a basis for E is given by the following seven sections of $\mathcal{O}(3)$:

$$\begin{aligned} &s_1 = x^2 y\,, \qquad s_2 = x y^2\,, \qquad s_3 = y^2 z\,, \\ &s_4 = y z^2\,, \qquad s_5 = x z^2\,, \qquad s_6 = x^2 z\,, \qquad s_7 = xyz\,. \end{aligned} \tag{3.2}$$

These seven sections determine a rational map $\mathbb{P}^2 \to \mathbb{P}^6$ via $[x : y : z] \to [s_1 : \ldots : s_7]$ (the map is only rational because the sections have common zeroes at P_1, P_2, and P_3). However, we can take the divisors on $\mathbb{P}^2$ corresponding to these seven sections and then blow them up to get seven effective divisors on X. These seven divisors correspond to seven sections (which we will also denote s_i) of a line-bundle on X, and these seven sections on X actually determine an embedding $X \hookrightarrow \mathbb{P}^6$. The image of X in $\mathbb{P}^6$ corresponds to the projective ideal $I \subset \mathbb{C}[s_1, s_2, s_3, s_4, s_5, s_6, s_7]$ generated by polynomial relations arising from (3.2). This ideal has a presentation with nine quadratic generators:

$$\begin{array}{lll} s_1 s_7 - s_2 s_6\,, & s_2 s_7 - s_1 s_3\,, & s_3 s_7 - s_2 s_4\,, \\ s_4 s_7 - s_3 s_5\,, & s_5 s_7 - s_4 s_6\,, & s_6 s_7 - s_1 s_5\,, \\ s_7^2 - s_1 s_4\;\;, & s_7^2 - s_2 s_5\;\;, & s_7^2 - s_3 s_6. \end{array} \tag{3.3}$$

So the affine coordinate ring of the cone over X is $A \cong \mathbb{C}[s_1, \ldots, s_7]/I$. In fact, we have seen the ring A before; the quadratic polynomials generating (3.3) are precisely the ones in (2.8) determining the chiral algebra of mesonic operators in the four models.

We are also interested in the symmetries of X, since they will be reflected in the symmetries of C and the associated horizon H. By symmetries, we mean the isometries of the Kähler-Einstein metric on X. Although the Kähler-Einstein metric on X is not known explicitly, we will assume that it respects the isometries of the Kähler metric on X induced from the Fubini-Study metric on $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (we could equally well consider the metric arising from $\mathbb{P}^6$). The only continuous isometries of X arise from the $\mathrm{U}(1)^2 \times \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$ which acts on $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Of this $\mathrm{U}(1)^5$, only a $\mathrm{U}(1)^2$ subgroup respects the equations (3.1) determining X. In terms of the embedding in $\mathbb{P}^6$, this $\mathrm{U}(1)^2$ can be taken to act with the following charges on the coordinates of $\mathbb{P}^6$:

$$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 \\ 0 & -1 & -1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 & -1 & 0 & 0 \end{pmatrix}. \tag{3.4}$$

As mentioned in section 2, X also has as a discrete symmetry the group D_{12}. The S_3 subgroup simply arises from permutations of the three points of the blowup on $\mathbb{P}^2$. In terms of the embedding in $\mathbb{P}^6$, it is generated by the following order-3 and order-2 permutations:

$$\begin{aligned} \mathbb{Z}_3:&\;\; (s_1 s_3 s_5)(s_4 s_6 s_2)\,, \\ \mathbb{Z}_2:&\;\; (s_1 s_3)(s_4 s_6) \end{aligned} \tag{3.5}$$

The other order-2 generator corresponds to the permutation

$$\mathbb{Z}_2:\;\; (s_1 s_4)(s_2 s_5)(s_3 s_6)\,. \tag{3.6}$$

We will see that this D_{12} is nicely realized on the set of six lines[7] on X.

We now discuss the divisors on X and its topology in general. X has three exceptional divisors corresponding to the three $\mathbb{P}^1$'s which are blown-up, which we will denote E_1, E_2, and E_3. Let h be a hyperplane in $\mathbb{P}^2$, and $\pi^* h = D$ the pullback to X. Then the Picard group of X is freely generated by D, E_1, E_2 and E_3, so it is isomorphic to $\mathbb{Z}^4$.

[7]By a line, we mean a $\mathbb{P}^1$ subvariety which arises from the intersection of hyperplanes in $\mathbb{P}^6$.

For this example, in fact, we have[8] $\mathrm{H}^2(X,\mathbb{Z}) \cong \operatorname{Pic} X$. The rest of the cohomology follows from the observation that π_1 is a birational invariant and from Poincare duality. So

$$\mathrm{H}_*(X) \cong \mathrm{H}^*(X,\mathbb{Z}) \cong \mathbb{Z},\, 0,\, \mathbb{Z}^4,\, 0,\, \mathbb{Z}\,. \tag{3.7}$$

The cup-product on H^* is determined by the intersections of divisors on X, which we may summarize as

$$D \cdot D = 1\,, \quad E_i \cdot E_j = -\delta_{ij}\,, \quad D \cdot E_i = 0\,. \tag{3.8}$$

As mentioned above, X has a set of six lines (these are related to the twenty-seven lines on the cubic surface in $\mathbb{P}^3$, which is also a del Pezzo surface). In terms of the divisors above, we can write the six lines as

$$\begin{aligned} L_1 &= D - E_2 - E_3\,, & L_2 &= D - E_1 - E_3\,, & L_3 &= D - E_1 - E_2\,, \\ L_4 &= E_1\,, & L_5 &= E_2\,, & L_6 &= E_3\,. \end{aligned} \tag{3.9}$$

The lines L_1, L_2, and L_3 correspond to the blow-ups of the lines on $\mathbb{P}^2$ passing though the points P_2 and P_3, P_1 and P_3, and P_1 and P_2 respectively. The other lines L_4, L_5, and L_6 are the exceptional divisors. In the homogeneous coordinates of $\mathbb{P}^6$, the six lines correspond to the following $\mathbb{P}^1$ subvarieties:

$$\begin{aligned} L_1 &= [0:0:s_3:s_4:0:0:0]\,, & L_4 &= [s_1:0:0:0:0:s_6:0]\,, \\ L_2 &= [0:0:0:0:s_5:s_6:0]\,, & L_5 &= [0:s_2:s_3:0:0:0:0]\,, \\ L_3 &= [s_1:s_2:0:0:0:0:0]\,, & L_6 &= [0:0:0:s_4:s_5:0:0]\,. \end{aligned} \tag{3.10}$$

The notation above (perhaps somewhat non-standard) just means that the lines correspond to the $\mathbb{P}^1$ subvarieties of $\mathbb{P}^6$ for which all homogeneous coordinates but the indicated pairs are zero.

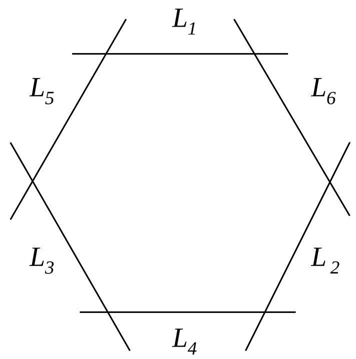

Figure 2: Six lines on dP_3.

We see from (3.8) that each line has self-intersection $L_i^2 = -1$. Each line also intersects transversely two other lines, as shown in figure 2. The action of D_{12} on the six lines is the natural one suggested by the figure.

These lines also provide another way of understanding the embedding $X \hookrightarrow \mathbb{P}^6$. We began with a linear system in $\mathcal{O}(3)$, corresponding to the divisor $3h$, on $\mathbb{P}^2$. Away from the points P_1, P_2, and P_3, h is naturally isomorphic to $\tilde{h}$, the effective divisor on X which blows-down to h. $\tilde{h}$ must then be the closure in X of $\pi^{-1}(h - P_1 - P_2 - P_3)$. In terms of $\operatorname{Pic} X$ we then naturally identify $\tilde{h} = D - E_1 - E_2 - E_3$. Similarly, the effective divisor on X arising from the blow-up of $3h$ is $-K = 3D - E_1 - E_2 - E_3$, the anti-canonical class of X. So the embedding $X \hookrightarrow \mathbb{P}^6$ arises from the anti-canonical bundle $\mathcal{O}(-K)$ on X. In terms of the six lines, the anti-canonical class of X is $-K = L_1 + \cdots + L_6$. Let $x_1, \ldots, x_6$ be generic global sections of the line-bundles $\mathcal{O}(L_1), \ldots, \mathcal{O}(L_6)$. Since $\mathcal{O}(L_1) \otimes \cdots \otimes \mathcal{O}(L_6) \cong \mathcal{O}(-K)$, we can write the sections s_i of $\mathcal{O}(-K)$ which we used to make the embedding in terms of the x_i. We

[8] One can also see this directly using the relative long-exact sequence for pair $(X, E_1 \cup E_2 \cup E_3)$.

make this identification in the obvious way. On $\mathbb{P}^2$ each section s_i corresponds to a divisor which is a linear combination of the lines through P_2 and P_3, P_1 and P_3, and P_1 and P_2. These divisors blow-up to the corresponding combinations of the lines L_1, L_2, and L_3 on X plus the combinations of the exceptional divisors L_4, L_5, and L_6 which are necessary so that the divisor is in the class of $-K$. For each linear combination of the lines L_i, we have a corresponding product of the sections x_i representing s_i in $\Gamma(X, \mathcal{O}(-K))$. Referring back to (3.2), we find expressions for the s_i in terms of the "homogeneous" x_i to be

$$\begin{aligned} &s_1 = x_1^2 x_2 x_5 x_6^2\,, \quad s_2 = x_1 x_2^2 x_4 x_6^2\,, \quad s_3 = x_2^2 x_3 x_4^2 x_6\,, \\ &s_4 = x_2 x_3^2 x_4^2 x_5\,, \quad s_5 = x_1 x_3^2 x_4 x_5^2\,, \quad s_6 = x_1^2 x_3 x_5^2 x_6\,, \quad s_7 = x_1 x_2 x_3 x_4 x_5 x_6\,. \end{aligned} \tag{3.11}$$

Note that indeed these assignments agree with the fact that x_i must vanish along the corresponding line L_i. We have made this rather extended discussion because it will be most natural to talk about the three-cycles on H which D3-branes wrap in terms of linear systems of divisors on X. We specify these divisors as the vanishing loci of sections of line-bundles on X, and we write these sections in terms of the x_i. For instance, we find it useful to record that the $\mathrm{U}(1)^2$ isometry of X may be considered to arise from a corresponding $\mathrm{U}(1)^2$ action on the x_i,

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ -2 & 0 & 0 & -1 & 1 & 2 \end{pmatrix} \end{array} . \tag{3.12}$$

On the other hand, we have a $\mathrm{U}(1)^4$ subgroup which acts non-trivially on the x_i but does not act on X (as one sees from (3.11)),

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array} . \tag{3.13}$$

3.2 The geometry and topology of H

Of slightly more direct interest is the geometry of the horizon H appearing as the azimuthal directions in the cone over X. Based on the AdS/CFT correspondence, we expect that the infrared dynamics of the gauge theories in section 2 are described by type IIB strings in the background $\mathrm{AdS}_5 \times H$ with N units of five-form flux through H [5]. In terms of our discussion of X, H is described as the S^1-bundle in the anti-canonical bundle $\mathcal{O}(-K)$ over X.

Since H is just an S^1-bundle over X, its cohomology is determined via the Leray spectral sequence [23] in terms of the cohomology of X. As discussed in [5], the homology and cohomology of H are

$$\mathrm{H}_*(H^5) \cong \mathrm{H}^*(H^5, \mathbb{Z}) \cong \mathbb{Z}, 0, \mathbb{Z}^3, \mathbb{Z}^3, 0, \mathbb{Z}\,. \tag{3.14}$$

The element of $\mathrm{H}^2(X, \mathbb{Z})$ which is sent to zero upon pullback to H is the Chern class of the S^1-bundle, which corresponds to the divisor $L_1 + \cdots + L_6$. Also, corresponding to the six lines on X, H has six three-cycles which wrap each $\mathbb{P}^1$ on the base and the S^1 in the fiber.

JHEP12(2001)001

In terms of the dual AdS theory, we will see the effect of this topology in two places. First, the three-cycles on H lead in the dual AdS theory to U(1) gauge symmetries arising from the reduction of the self-dual R-R four-form along them. These gauge symmetries will appear on the worldvolume as global U(1) symmetries not corresponding to isometries of H. Since $\mathrm{H}_3(H) \cong \mathbb{Z}^3$, we expect a $\mathrm{U}(1)^3$ symmetry to arise in this fashion, in agreements with what we found in section 2. Second, the three-cycles of H also provide for the existence of states in the gauge theory corresponding to wrapped D3-branes. These D3-brane states will be charged under the $\mathrm{U}(1)^3$ arising from the corresponding three-cycles, and so the $\mathrm{U}(1)^3$ should appear as a baryonic symmetry in the gauge theories — indeed, we observed in section 2 this symmetry as $\mathrm{U}(1)_C \times \mathrm{U}(1)_D \times \mathrm{U}(1)_E$.

The isometries of H are also related to those of X. (Again, we do not know the Einstein metric on H, so we look at the isometries of the metric on H induced from the Euclidean metric in $\mathbb{C}^7$.) The $\mathrm{U}(1)^2$ isometry of (3.4) lifts to H with a trivial action on the S^1 fibre. Having already matched the mesonic operators in the gauge theories to the coordinates s_i of $\mathbb{C}^7$, we see that indeed the action (3.4) corresponds to the action of $\mathrm{U}(1)_A \times \mathrm{U}(1)_B$ on mesonic operators in (2.9). On the other hand, H has an additional U(1) isometry which arises from rotations in the S^1 fibre alone. This isometry is the restriction to H of the U(1) action on the cone C under which the coordinates s_i all have unit charge — which we recognize in the gauge theories as the $\mathcal{R}$-symmetry. In terms of the sections x_i, this symmetry corresponds to the U(1) in (3.13) under which all x_i have unit charge. The fact that this U(1) — which does not act on X — is promoted to a symmetry of H corresponds to the fact that $L_1 + \cdots + L_6$ becomes topologically trivial on H and hence should not lead to a baryonic symmetry in the gauge theories. The remaining $\mathrm{U}(1)^3$ in (3.13) which acts on the x_i but not H corresponds to the baryonic symmetry in the gauge theories.

Finally, the discrete D_{12} symmetry of X lifts also to H with a trivial action on the fibre. The U(1) isometry of H corresponding to rotations in the fibre over X is distinguished by the fact that it is the only subgroup of $\mathrm{U}(1)^3$ which commutes with the D_{12} (we similarly observe that in the gauge theory, only $\mathrm{U}(1)_{\mathcal{R}}$ in $\mathrm{U}(1)_A \times \mathrm{U}(1)_B \times \mathrm{U}(1)_{\mathcal{R}}$ commutes with the D_{12} action in Models I and IV). So we identify the isometry group of H as

$$G = \mathrm{U}(1)^3 \rtimes D_{12} \,. \tag{3.15}$$

Except for the discrete symmetry, we have a direct correspondence between symmetries in the four gauge theories and the geometry of H.

We close this section by applying the method of [17] to compute the volume of H. The Einstein-Sasaki metric on H takes the form [24]

$$\mathbf{g} = \pi^* \mathbf{h} + \eta \otimes \eta \,, \tag{3.16}$$

where $\mathbf{h}$ is the Kähler-Einstein metric on X, π is the projection in the S^1-bundle from H onto X, and η is a connection 1-form on the fibration with curvature $d\eta = 2\pi^*\omega$ (ω the Kähler form on X). (More fundamentally, η is the 1-form dual to the distinguished vector field on H generating the regular U(1) isometry whose orbits form the fibres of the bundle $H \to X$.) Since the metric on H decomposes as a direct sum on the base X and the S^1 fibre, it suffices to compute separately the volume of X and the length of the fibre.

JHEP12(2001)001

The volume of X is determined using the Einstein condition, which relates the Kähler form to the first Chern class of X (we refer the reader to [17] for a careful exposition of the ubiquitous constants below),

$$\omega = \frac{\pi}{3} c_1(X)\,. \tag{3.17}$$

So, using the volume form $\frac{1}{2}\omega^2$, we see that

$$\mathrm{Vol}(X) = \frac{\pi^2}{18}\int_X c_1(X)^2\,. \tag{3.18}$$

In terms of $\mathrm{H^2(X,\mathbb{Z})}$, $c_1(X) = -K = 3D - E_1 - E_2 - E_3$. So the cup product (3.8) determines $\int_X c_1(X)^2 = 6$, hence $\mathrm{Vol}(X) = \pi^2/3$.

To determine the length of the fiber, we write $\eta = d\phi - \sigma$, where ϕ is an angular coordinate in the fiber and $d\sigma = 2\omega = (2\pi/3)\,c_1(X)$. Then the fiber has length 2π exactly when $d\sigma = 2\pi\, e$, where e is the Euler class (or equivalently first Chern class) of the S^1-bundle over X. In our case, $e = c_1(X)$, so (upon rescaling) we see that the length of the fiber is actually $2\pi/3$.

Hence the volume of H is $\mathrm{Vol}(H) = (2\pi/3)\ \mathrm{Vol}(X) = 2\pi^3/9$. For the $\mathbb{C}^3/(\mathbb{Z}_3\times\mathbb{Z}_3)$ orbifold, with horizon $S^5/(\mathbb{Z}_3\times\mathbb{Z}_3)$, the volume of S^5 in the Einstein metric is π^3, so that $\mathrm{Vol}(S^5/(\mathbb{Z}_3\times\mathbb{Z}_3)) = \pi^3/9$. Thus, $\mathrm{Vol}(S^5/(\mathbb{Z}_3\times\mathbb{Z}_3)) : \mathrm{Vol}(H) = 1:2$ as claimed in section 2.

We also make the following observation, relevant for our discussion of wrapped D3-branes in section 4. Suppose that $\mathcal{L}$ is a three-cycle on H which wraps the S^1 in the fibre and a holomorphic curve C on X. Then we can compute $\mathrm{Vol}(\mathcal{L}) = (2\pi/3)\,\mathrm{Vol}(C)$ as above. Since C is holomorphic, $\mathrm{Vol}(C) = \int_C \omega = \omega(C)$, which is determined by our knowledge of the class of ω from (3.17) and the class of C.

In fact, we note that three-cycles of this form are the supersymmetric ones, i.e. they have minimal volume among all three-cycles in the same homology class. To see this, note that any non-trivial three-cycle on H must wrap the S^1 in the fibre and hence projects to some two-dimensional submanifold of X. Because the Einstein metric on H is a direct sum on X and the fibre, it suffices to consider the volume of the two-manifold on X. But it is well-known that on any Kähler manifold, the holomorphic submanifolds are volume-minimizing within their respective homology classes (they are calibrated by ω). Thus, for D3-branes wrapping supersymmetric three-cycles, we can compare the volumes of the wrapped three-cycles (or effectively the corresponding holomorphic curves C on X, since we are only interested in volume ratios) to the $\mathcal{R}$-charges of the corresponding operators in the gauge theory.

4. Baryons and wrapped D3-branes

In this section, we will use the $\mathcal{N} = 1$ duality to determine the quantum algebra of baryonic chiral operators in the four models, at least for operators of $\mathcal{R}$-charge up to N. The full chiral algebra of operators in each model corresponds to the algebra of functions on moduli space of the theory. A simple-minded calculation of this part of the chiral ring leads to results that differ among the various models. We will show that the chiral algebras do become isomorphic if we include in each model quantum relations which arise under $\mathcal{N}=1$ duality from classical relations in the duals (similar to the story in [12]). We will further motivate the resulting baryonic spectrum by showing that it has a natural interpretation in terms of wrapped D3-branes in the dual AdS theory (similar to results in [25, 5]).

4.1 The quantum ring of baryons

We have already recorded in the appendices the relevant classical baryonic generators. The $\mathcal{N}=1$ duality, along with the requirement that corresponding operators in dual theories must transform in the same representations under all global symmetries, determines the correspondence between the various baryons in the four models. This correspondence between baryonic operators in the four models is indicated by corresponding labels $\mathcal{B}_i$, $\mathcal{B}_i(s)$, and $\mathcal{B}_i(s,t)$ in the appendices.

A calculation shows, however, that the classically computed chiral rings of the four models will not be isomorphic. If we wish to claim that they are dual, even in the weak sense of describing different points on the same critical manifold, we will have to resolve this puzzle. The subtlety here [12] is that the classical ring may be modified by "quantum relations", extra identifications imposed by the quantum dynamics. In this case, as in the case studied in [12], the quantum relations arise as *classical* relations in a dual theory. These relations arise when a composite operator $M_{ij}=X_{ik}X_{kj}$ in the direct model appears as a fundamental field X_{ij} in the dual model (e.g. (2.19)). Then, in the direct model the classical relation

$$M_{ij}^N = X_{ik}^N X_{kj}^N \tag{4.1}$$

holds. The dual model has dual baryonic operators $\tilde{X}_{kr}^N$, $\tilde{X}_{sk}^N$ corresponding to the direct baryons X_{ik}^N, X_{kj}^N, and similarly X_{ij}^N corresponding to M_{ij}^N. But the classical algebra of the dual model lacks the relation (henceforth we will be schematic, putting aside the issue of normalizations and scales which must also enter such relations)

$$X_{ij}^N = \tilde{X}_{kr}^N \tilde{X}_{sk}^N \tag{4.2}$$

corresponding to the relation (4.1). In our models, we propose such quantum relations to fix the operator algebras.

Since we have four models all related by chains of $\mathcal{N}=1$ dualities, if the rings of baryonic operators are to be isomorphic we must require that all relations (quantum or classical) appearing in a given model correspond to relations in the dual models. Since one generically finds new quantum relations each time one takes the dual, one might wonder at the start whether all quantum relations can be consistently imposed between all four models. We now show that a set of consistent relations can be imposed in all four models, and the resulting algebra has a very nice geometric interpretation.

To start in Model I, there is one fundamental field X_{54} which appears as a composite field in Model II — we also see this fact in the correspondence between baryonic operators $X_{54}^s(X_{53}X_{34})^{N-s} \leftrightarrow (X_{51}X_{14})^s(X_{53}X_{34})^{N-s}$ under the global symmetries. So we expect quantum relations in Model I involving baryonic operators constructed from X_{54}. Because

Model I has a D_{12} symmetry which should be preserved, we must symmetrize any set of quantum relations we impose. We then conjecture the following quantum relations in Model I:

$$\begin{aligned} X_{32}^s(X_{34}X_{42})^{N-s} &= (X_{16}X_{65})^s X_{15}^{N-s} \\ X_{26}^s(X_{21}X_{16})^{N-s} &= (X_{53}X_{34})^s X_{54}^{N-s} \\ X_{63}^s(X_{65}X_{53})^{N-s} &= (X_{42}X_{21})^s X_{41}^{N-s} \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} (X_{65}X_{54})^s(X_{63}X_{34})^{t-s}(X_{65}X_{53}X_{34})^{N-t} &= (X_{32}X_{21})^s(X_{34}X_{42}X_{21})^{t-s}(X_{34}X_{41})^{N-t} \\ &= (X_{21}X_{16}X_{65})^s(X_{21}X_{15})^{t-s}(X_{26}X_{65})^{N-t} \\ (X_{41}X_{16})^s(X_{42}X_{26})^{t-s}(X_{42}X_{21}X_{16})^{N-t} &= (X_{53}X_{32})^s(X_{53}X_{34}X_{42})^{t-s}(X_{54}X_{42})^{N-t} \\ &= (X_{16}X_{65}X_{53})^s(X_{15}X_{53})^{t-s}(X_{16}X_{63})^{N-t} . \end{aligned} \tag{4.4}$$

Note that these relations are consistent with all global symmetries. We have found in the quantum ring six generators of $\mathcal{R}$-charge $\frac{1}{3}N$,

$$\mathcal{B}_1 , \quad \mathcal{B}_2 , \quad \mathcal{B}_3 , \quad \mathcal{B}_4 , \quad \mathcal{B}_5 , \quad \mathcal{B}_6 , \tag{4.5}$$

three families of generators of $\mathcal{R}$-charge $\frac{2}{3}N$,

$$\mathcal{B}_1(s) , \quad \mathcal{B}_2(s) , \quad \mathcal{B}_3(s) , \tag{4.6}$$

and two families of generators of $\mathcal{R}$-charge N,

$$\mathcal{B}_1(s,t) , \quad \mathcal{B}_2(s,t) . \tag{4.7}$$

Now, dualizing on the first $\mathrm{SU}(N)$ factor implies that three fundamental fields of Model II (i.e. Y_{26}, X_{25}, Y_{46}) are composites in Model I. So quantum relations in Model II arise involving baryonic operators constructed from these fields. The other quantum relations (4.3), (4.4) appearing in Model I must also have their counterparts in Model II. So together we find the following relations in Model II:

$$\begin{aligned} X_{32}^s(X_{34}X_{42})^{N-s} &= (X_{65}X_{51})^s Y_{61}^{N-s} , \\ X_{26}^s Y_{26}^{N-s} &= (X_{53}X_{34})^s (X_{51}X_{14})^{N-s} , \\ X_{63}^s(X_{65}X_{53})^{N-s} &= (X_{14}X_{42})^s X_{12}^{N-s} , \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} X_{25}^s(X_{26}X_{65})^{t-s}(Y_{26}X_{65})^{N-t} &= ((X_{34}X_{42})X_{14})^s(X_{34}X_{12})^{t-s}(X_{32}X_{14})^{N-t} , \\ &= (Y_{61}X_{14})^s(X_{65}X_{53}X_{34})^{N-t}(X_{65}X_{51}X_{14})^{N-t} , \\ Y_{46}^s(X_{42}X_{26})^{t-s}(X_{42}Y_{26})^{N-t} &= (X_{51}X_{12})^s(X_{53}X_{34}X_{42})^{t-s}(X_{51}X_{14}X_{42})^{N-t} , \\ &= (X_{51}(X_{65}X_{53}))^s(Y_{61}X_{53})^{t-s}(X_{51}X_{63})^{N-t} . \end{aligned} \tag{4.9}$$

We see the same families of generators as in (4.5), (4.6), and (4.7).

Now if we consider Model II as dual to Model III, we have two fundamental fields in Model II which appear as composites in Model III, namely X_{32} and X_{12}. However, the relations implied from Model III already appear above as coming from Model I. On the other hand, in Model III there are two fundamental fields X_{61} and Y_{63} which appear as composites in Model II, from which more quantum relations arise. To these relations, we must also add the other relations which appeared in Model II. So we find as quantum relations in Model III

$$\begin{aligned} X_{61}^s Y_{61}^{N-s} &= (X_{35}X_{52})^s (X_{34}X_{42})^{N-s}\,, \\ X_{26}^s Y_{26}^{N-s} &= (X_{15}X_{34})^s (X_{35}X_{14})^{N-s}\,, \\ X_{63}^s Y_{63}^{N-s} &= (X_{14}X_{42})^s (X_{15}X_{52})^{N-s}\,, \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} X_{56}^s (X_{52}X_{26})^{t-s} (X_{52}Y_{26})^{N-t} &= ((X_{34}X_{42})X_{14})^s ((X_{15}X_{52})X_{34})^{t-s} ((X_{35}X_{52})X_{14})^{N-t}\,, \\ &= (Y_{61}X_{14})^s (Y_{63}X_{34})^{t-s} (X_{61}X_{14})^{N-t}\,, \\ Y_{46}^s (X_{42}X_{26})^{t-s} (X_{42}Y_{26})^{N-t} &= (X_{15}(X_{35}X_{52}))^s (X_{15}(X_{34}X_{42}))^{t-s} (X_{35}(X_{14}X_{42}))^{N-t}\,, \\ &= (X_{61}X_{15})^s (Y_{61}X_{15})^{t-s} (X_{63}X_{35})^{N-t}\,. \end{aligned} \tag{4.11}$$

These relations prune the baryonic operator algebra to the same families of generators as in Models I and II.

Finally, note that none of the fields of Model III correspond to composites of Model IV (all those fields were massive and integrated out after dualizing), so there are no new quantum relations in Model III arising from classical relations in Model IV. In Model IV there are four fundamental fields appearing as composites in Model III (i.e. X_{46}, Z_{46}, Y_{56}, and Z_{56}). So besides the quantum relations which carry over from Model III, we predict new quantum relations in Model IV involving these fields. We conjecture the relations

$$\begin{aligned} X_{61}^s Y_{61}^{t-s} &= (X_{35}X_{24})^s (X_{34}X_{25})^{t-s}\,, \\ X_{62}^s Y_{62}^{t-s} &= (X_{15}X_{34})^s (X_{35}X_{14})^{t-s}\,, \\ X_{63}^s Y_{63}^{t-s} &= (X_{14}X_{25})^s (X_{15}X_{24})^{t-s}\,, \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} X_{46}^s Y_{46}^{t-s} Z_{46}^{N-t} &= (X_{15}X_{34}X_{25})^s (X_{15}X_{35}X_{24})^{t-s} (X_{35}X_{14}X_{25})^{N-t}\,, \\ &= (X_{62}X_{25})^s (X_{61}X_{15})^{t-s} (Y_{62}X_{25})^{N-t}\,, \\[1em] X_{56}^s Y_{56}^{t-s} Z_{56}^{N-t} &= (X_{34}X_{25}X_{14})^s (X_{34}X_{15}X_{24})^{t-s} (X_{35}X_{24}X_{14})^{N-t}\,, \\ &= (Y_{61}X_{14})^s (X_{62}X_{24})^{t-s} (Y_{62}X_{24})^{N-t}\,. \end{aligned} \tag{4.13}$$

So again we see the same families of generators, and we have consistently and simultaneously imposed in all four models all quantum relations we expect to arise under the $\mathcal{N}=1$ dualities.

4.2 Wrapped D3-branes on H

We now explain the appearance of the six baryonic generators of $\mathcal{R}$-charge $\frac{1}{3}N$, the three families of baryonic generators of $\mathcal{R}$-charge $\frac{2}{3}N$, and the two families of $\mathcal{R}$-charge N in terms of wrapped D3-branes on H. Recall that the $\mathcal{R}$-charge of the operator in the gauge theory determines its conformal weight at the conformal point, and hence the mass of the corresponding state in the AdS theory. So we know that the ratios of the masses of the corresponding brane states must be $1:2:3$. The mass of the wrapped brane will be proportional to the volume of the wrapped cycle. So we must look for families of supersymmetric three-cycles having volume ratio $1:2:3$ on H. Equally important, we know the action of the global symmetries $\mathrm{U}(1)^5 \rtimes D_{12}$ on the baryonic generators, and, since we know how to interpret these symmetries in the AdS theory, we should find the same action on the wrapped-brane states.

Consider first the six baryonic generators $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$, $\mathcal{B}_5$, and $\mathcal{B}_6$ of $\mathcal{R}$-charge $\frac{1}{3}N$ (for which we can take representatives in Model I X_{21}^N, X_{65}^N, X_{34}^N, X_{53}^N, X_{42}^N, and X_{16}^N). The D_{12} symmetry acts on these operators as it acts on the six lines L_i on X, and hence it acts on these operators as it acts on the six three-cycles in H wrapping the lines and the fibre. We hypothesize then that these baryons correspond to D3-branes wrapping these three-cycles. As we indicated in section 3, the supersymmetric three-cycles on H will arise in this way, wrapping a divisor on X and the S^1 fibre. To simplify notation, we will refer both to the divisors on X and the three-cycle on H by means of the sections x_i (corresponding to the lines L_i) introduced in section 3. So we propose the following correspondence of baryons and D3-branes wrapping the following three-cycles:

$$\begin{aligned} &\mathcal{B}_1 \sim x_1\,, \qquad &&\mathcal{B}_4 \sim x_4\,, \\ &\mathcal{B}_2 \sim x_2\,, \qquad &&\mathcal{B}_5 \sim x_5\,, \\ &\mathcal{B}_3 \sim x_3\,, \qquad &&\mathcal{B}_6 \sim x_6\,. \end{aligned} \tag{4.14}$$

Since the lines L_i are all degree 1 curves on X, the S^1 in H fibres over them via the usual Hopf fibration. So the corresponding three-cycles on H are topologically S^3's. The volumes of these three-cycles on H are determined by the volumes of the curves L_i on X, as discussed in section 3. We see that the volumes of these curves are $\mathrm{Vol}(L_i) = \omega(L_i) = \frac{\pi}{3}\,(-K\cdot L_i) = \frac{\pi}{3}$. (Incidentally, $-K\cdot C$ just determines the degree of the curve C on X.)

We can also check that the $\mathrm{U}(1)^5$ action on these three-cycles agrees with the corresponding action on the baryonic states. First, the correspondence of baryons and wrapped branes (4.14) implies that the $\mathrm{U}(1)_A \times \mathrm{U}(1)_B$ symmetry of the gauge theory corresponds to a $\mathrm{U}(1)^2$ action on the sections x_i over X,

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}. \tag{4.15}$$

Comparing to (3.12), we see that the $\mathrm{U}(1)^2$ expected from the geometry and the $\mathrm{U}(1)^2$ discovered in the gauge theory are the same.

The other $\mathrm{U}(1)_C \times \mathrm{U}(1)_D \times \mathrm{U}(1)_E$ subgroup is a baryonic symmetry under which all mesons are uncharged and which does not correspond to a geometric symmetry of H. With

JHEP12(2001)001

the correspondence (4.14), the $\mathrm{U}(1)_C \times \mathrm{U}(1)_D \times \mathrm{U}(1)_E$ symmetry corresponds to a $\mathrm{U}(1)^3$ action on the sections x_i with charges

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \end{pmatrix} . \tag{4.16}$$

We see that this $\mathrm{U}(1)^3$ baryonic symmetry is precisely the $\mathrm{U}(1)^3$ subgroup of the $\mathrm{U}(1)^4$ action in (3.13) which we predicted to appear. Of course, the baryonic operators are homogeneously charged under the $\mathrm{U}(1)_{\mathcal{R}}$ symmetry, which we have explained corresponds to the homogeneous action of the last U(1) subgroup of (3.13).

We now consider baryonic operators of $\mathcal{R}$-charge $\frac{2}{3}N$. There are three families of operators, $\mathcal{B}_1(s)$, $\mathcal{B}_2(s)$, and $\mathcal{B}_3(s)$ (for which we can consider representatives $X_{32}^s(X_{34}X_{42})^{N-s}$, $X_{62}^s(X_{21}X_{16})^{N-s}$, and $X_{63}^s(X_{65}X_{53})^{N-s}$ in Model I). Under the D_{12} symmetry, the S_3 subgroup permutes the three families. In the quantum ring, for which (4.3) holds, the remaining $\mathbb{Z}_2$ permutes the baryons within the three families, acting by $s \leftrightarrow N-s$. So we look for three families of three-cycles permuted by the S_3 subgroup and fixed by the $\mathbb{Z}_2$. Based on the $\mathcal{R}$-symmetry, we also look for families of three-cycles for which each member has twice the volume of the isolated three-cycles associated to the baryons of $\mathcal{R}$-charge $\frac{1}{3}N$. Finally, we note that in each family of operators $\mathcal{B}_i(s)$ for $0 \leq s \leq N$, the operators $\mathcal{B}_i(s=0)$ and $\mathcal{B}_i(s=N)$ are distinguished by the fact that they are not really generators in the ring but occur as products of $\mathcal{R}$-charge $\frac{1}{3}N$ baryons. E.g. in Model I, $\mathcal{B}_1(s=0) = X_{34}^N X_{42}^N = \mathcal{B}_3\mathcal{B}_5$, and similarly for $\mathcal{B}_1(s=N) = X_{32}^N = X_{16}^N X_{65}^N = \mathcal{B}_6\mathcal{B}_2$ (note that the first relation for $\mathcal{B}_1(s=0)$ is classical, but the relation for $\mathcal{B}_1(s=N)$ is a quantum relation in Model I).

These observations lead us to make the following hypothesis for the correspondence of these baryonic operators to D3-branes wrapping moduli of three-cycles on H. First, note that the six lines on X are of course not all independent in homology, and we have relations in homology between pairs of lines on X

$$\begin{aligned} L_2 + L_6 &= L_3 + L_5 = D - E_1\,, \\ L_3 + L_4 &= L_1 + L_6 = D - E_2\,, \\ L_1 + L_5 &= L_2 + L_4 = D - E_3\,, \end{aligned} \tag{4.17}$$

which imply corresponding relations between three-cycles on H. So we can consider three linear systems of curves on X in the same divisor classes as the sums of lines in (4.17). In terms of the sections x_i on X, these linear systems correspond to the following linear combinations of global sections of $\mathcal{O}(D-E_1)$, $\mathcal{O}(D-E_2)$, and $\mathcal{O}(D-E_3)$:

$$a\,x_2x_6 + b\,x_3x_5\,, \qquad a\,x_3x_4 + b\,x_1x_6\,, \qquad a\,x_1x_5 + b\,x_2x_4\,. \tag{4.18}$$

For each such system of curves on X, there is a corresponding family of three-cycles on H which wrap these curves on X and the S^1 in the fibre. The volumes of these three-cycles are determined by the volumes of the curves $(D - E_i)$ on X, which we compute as $\mathrm{Vol}(D - E_i) = \frac{\pi}{3}(-K \cdot (D - E_i)) = \frac{2\pi}{3}$, twice the volume of the isolated curves for baryons of $\mathcal{R}$-charge $\frac{1}{3}N$.

We then conjecture that the correspondence between the three families of baryonic operators and the corresponding three-cycles over these divisors is

$$\begin{aligned}
\mathcal{B}_1(s) &\sim a\, x_2 x_6 + b\, x_3 x_5\,, \\
\mathcal{B}_2(s) &\sim a\, x_3 x_4 + b\, x_1 x_6\,, \\
\mathcal{B}_3(s) &\sim a\, x_1 x_5 + b\, x_2 x_4\,.
\end{aligned} \tag{4.19}$$

Here, the parameters a and b are coefficients specifying a point in a $\mathbb{P}^1$ moduli space of three-cycles. We see immediately that the S_3 symmetry of H permutes these three families of three-cycles, and the $\mathbb{Z}_2$ acts on the $\mathbb{P}^1$ moduli by $[a : b] \mapsto [b : a]$. So the D_{12} action on these three-cycles corresponds to the same action on the three families of baryons.

In order to discuss the $\mathrm{U}(1)^6$ action on the wrapped-brane states, we will first discuss the $\mathbb{P}^1$ moduli in more detail. At the poles $[1 : 0]$ and $[0 : 1]$ of $\mathbb{P}^1$, the corresponding curves in the linear systems in (4.18) are reducible, and the branes wrap a pair of the individual S^3's corresponding to baryons of $\mathcal{R}$-charge $\frac{1}{3}N$. On X the corresponding divisors intersect transversely, so the S^3's intersect along the S^1 in the fiber. The reducibility of these curves on X corresponds to the factorization of the distinguished baryonic operators $\mathcal{B}_i(s = 0)$ and $\mathcal{B}_i(s = N)$.

At generic points in the $\mathbb{P}^1$ moduli space, the corresponding three-cycle is topologically $\mathbb{RP}^3$. One can see this by noting that the zero-locus of a section $a\, x_2 x_6 + b\, x_3 x_5$ for generic a, b, is an irreducible curve of genus zero (this follows from the adjunction formula), hence topologically S^2. However, this curve has degree two in $\mathbb{P}^6$, which means that the S^1 fibration has twice the Chern class of the usual Hopf fibration — implying that the three-cycle on H is topologically $\mathbb{RP}^3$.

The point of this discussion is that the moduli space of the three-cycles about which these D3-branes wrap is not merely $\mathbb{P}^1$, but $\mathbb{P}^1$ with two distinguished points (the poles). Although one might have expected the full $\mathrm{SU}(2)$ symmetry of $\mathbb{P}^1$ to act on the moduli, because of the distinguished points only a $\mathrm{U}(1)$ acts. Now, as explained in [25], the dynamics of the wrapped D3-branes will be described by quantum mechanics on the moduli space of wrapped three-cycles. Because the D3-branes are electrically charged with respect to the N units of 5-flux, the D3-brane wavefunction will not be a function on $\mathbb{P}^1$ but a global section of the line-bundle $\mathcal{O}(N)$, and this bundle has $N + 1$ sections transforming with charges $0, \ldots, N$ under the $\mathrm{U}(1)$ symmetry.

This observation explains the degeneracies within the three families of baryons. First, the action of the $\mathrm{U}(1)_C \times \mathrm{U}(1)_D \times \mathrm{U}(1)_E$ symmetry on the baryonic operators in the gauge theory is consistent with the associated $\mathrm{U}(1)^3$ action induced from (4.16) on the divisors in (4.19). The $\mathrm{U}(1)_A \times \mathrm{U}(1)_B$ geometric symmetry is more interesting. In each family $\mathcal{B}_i(s)$ of baryons, under one $\mathrm{U}(1)$ generator of this symmetry all baryons are charged equally, and under the other $\mathrm{U}(1)$ generator the baryons within a given family have charges $0, 1, \ldots, N-1, N$. Comparing the symmetries $\mathrm{U}(1)_A \times \mathrm{U}(1)_B$ in table 1 to the corresponding

$\mathrm{U}(1)^2$ action on H in (4.15), we see that the $\mathrm{U}(1) \subset \mathrm{U}(1)_A \times \mathrm{U}(1)_B$ for which all baryons of a family are homogeneously charged, acts on H so that all three-cycles within a given family transform homogeneously (which is to say, it does not act on the $\mathbb{P}^1$ moduli). In contrast, the $\mathrm{U}(1) \subset \mathrm{U}(1)_A \times \mathrm{U}(1)_B$ for which the baryons in a given family have charges $0, \ldots, N$ corresponds to a $\mathrm{U}(1)$ action on H which induces the $\mathrm{U}(1)$ action on the $\mathbb{P}^1$ moduli. So the baryon charges of $0, \ldots, N$ under this symmetry are in agreement with the charges of the $N+1$ sections of $\mathcal{O}(N)$ under the $\mathrm{U}(1)$ symmetry of $\mathbb{P}^1$.

Turning to the baryons of $\mathcal{R}$-charge N, the story is very similar. In Model I we take representatives of the two families of baryons $\mathcal{B}_1(s,t)$ and $\mathcal{B}_2(s,t)$ to be $(X_{65}X_{54})^s \times (X_{63}X_{34})^{t-s}(X_{65}X_{53}X_{34})^{N-t}$ and $(X_{41}X_{16})^s(X_{42}X_{26})^{t-s}(X_{42}X_{21}X_{16})^{N-t}$ respectively. In this case the S_3 subgroup of the D_{12} symmetry of Model I permutes baryons within the two families, and the $\mathbb{Z}_2$ subgroup exchanges the two families. Further, we note that the baryons in each family for which $s = t = 0$, $s = N - t = 0$, or $s = t = N$ factor into the product of three $\mathcal{R}$-charge $\frac{1}{3}N$ baryons. Similarly, the baryons for which $s = 0$, $t - s = 0$, or $t = N$ factorize into the product of a $\mathcal{R}$-charge $\frac{1}{3}N$ baryon and a $\mathcal{R}$-charge $\frac{2}{3}N$ baryon.

Based on the discrete and global symmetries, we propose that the baryon-brane correspondence is given by the following linear combinations of global sections of $\mathcal{O}(2D - E_1 - E_2 - E_3)$ and $\mathcal{O}(D)$:

$$\begin{aligned} \mathcal{B}_1(s,t) &\sim (a\,x_1x_3x_5 + b\,x_2x_3x_4 + c\,x_1x_2x_6)\,, \\ \mathcal{B}_2(s,t) &\sim (a\,x_3x_4x_5 + b\,x_2x_4x_6 + c\,x_1x_5x_6)\,. \end{aligned} \tag{4.20}$$

Here $[a : b : c]$ are coordinates on a $\mathbb{P}^2$ moduli space for each family of three-cycles. We compute the volumes of the curves $(2D - E_1 - E_2 - E_3)$ and D on X to be $\mathrm{Vol}(2D - E_1 - E_2 - E_3) = \mathrm{Vol}(D) = \pi$, so these have the proper volume to represent baryons of $\mathcal{R}$-charge N.

We easily see that the action of the D_{12} and $\mathrm{U}(1)_C \times \mathrm{U}(1)_D \times \mathrm{U}(1)_E$ symmetries in the $\mathcal{N} = 1$ theory is consistent with the assignment (4.20). To understand the action of $\mathrm{U}(1)_A \times \mathrm{U}(1)_B$ on these baryons, we study more carefully the structure of the moduli space of three-cycles above. For instance, at the points $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$ on the $\mathbb{P}^2$, we see from figure 2 that the representative three-cycle is a chain of three volume-1 S^3's intersecting in the S^1 fibers over two points on X. We see that the factorization of these curves corresponds to the factorization of baryonic operators in the chiral algebra. We also see that along the three $\mathbb{P}^1$'s bounding the $\mathbb{P}^2$ moduli, i.e. $[a : b : 0]$, $[a : 0 : c]$, and $[0 : b : c]$, the representative three-cycle is one of the individual, volume-1 S^3's intersecting an $\mathbb{RP}^3$ from the volume-2 family along the S^1 in the fiber. Again, the baryonic operators have a corresponding factorization in the quantum chiral algebra. Finally, over a generic point in the $\mathbb{P}^2$ moduli the representative three-cycle is topologically $S^3/\mathbb{Z}_3$, the $\mathbb{Z}_3$ acting homogeneously on the embedding coordinates $S^3 \hookrightarrow \mathbb{C}^2$. (This fact follows because the corresponding curves on X are again genus zero but are of degree three.) Because of the distinguished points in the $\mathbb{P}^2$ moduli space, the symmetries that act on the moduli space are reduced from $PGL(3, \mathbb{C})$ to only $\mathrm{U}(1) \times \mathrm{U}(1)$. As before, the presence of N units of 5-flux implies that the D3-brane wavefunctions on this moduli space are sections of $\mathcal{O}(N)$

on $\mathbb{P}^2$. The global sections of $\mathcal{O}(N)$ on $\mathbb{P}^2$ are in 1-1 correspondence with the baryons in each family, and the geometric $\mathrm{U}(1)_A \times \mathrm{U}(1)_B$ symmetry of the gauge theory acts in accordance with the induced action of the $\mathrm{U}(1)^2$ symmetry in (4.15) on the $\mathbb{P}^2$ moduli.

5. Other toric dualities

We conclude by discussing the other examples of toric duality mentioned in [15]. These examples also arise from the $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ orbifold and correspond to cones over a del Pezzo surface dP_2 and a Hirzebruch surface $\mathbb{F}_0$. dP_2 is a complex surface obtained by blowing-up $\mathbb{P}^2$ at two generic points — hence it can also be obtained from dP_3 by blowing-down one of the exceptional divisors. As remarked earlier, $\mathbb{F}_0$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. In each of these cases, upon resolving the orbifold we find that two distinct gauge theories arise to describe the geometric phases corresponding to the complex cone over dP_2 and the cone over $\mathbb{F}_0$.

Since the cone over dP_2 is a resolution of the cone over dP_3, there should exist in the four dP_3 models relevant deformations which realize the corresponding dP_2 models. These deformations of course arise from Fayet-Iliopoulos terms in the dP_3 models. As usual, non-zero Fayet-Iliopoulos terms in the dP_3 models imply that some chiral fields get non-zero vevs and partially break the gauge symmetry. In order to see how to realize the dP_2 models from the dP_3 models (other than bothering with the sort of toric algorithms in [9]), we just need to know which chiral fields in the dP_3 models should get non-zero vevs.

Based on our baryon/brane correspondence, we can very easily see which chiral fields in the dP_3 models must get non-zero vevs to realize the dP_2 models. Note that, in terms of the gauge-invariant chiral operators, we only want to turn on vevs for baryonic operators, since giving non-zero vevs to mesonic operators will move the D3-branes away from the singularity. Now, when a baryonic operator in the dP_3 gauge theory gets a vev, a vanishing two-cycle in the cone over dP_3 is blown-up to non-zero volume.[9] In fact, to resolve the cone over dP_3 to the cone over dP_2, we want to blow-up any one of six lines on dP_3 (the D_{12} symmetry implies we needn't care which line). D3-branes wrapping the corresponding three-cycles on H^5 appear in the gauge-theories as the six baryonic operators of $\mathcal{R}$-charge $\frac{1}{3}N$. So we see that in order to realize the dP_2 models from the dP_3 models, we just need to turn on a vev for any one of the six chiral fields of $\mathcal{R}$-charge $1/3$.

So we first expect that the dP_2 models should have gauge group $\mathrm{SU}(N)^5$ (since only one vev is turned on), which also follows directly from the toric algorithm. Let us record here the superpotentials (also appearing in [15]) of the two dP_2 models which arise from the $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ orbifold (we use the same notation for bifundamental chiral matter as in the dP_3 models, so we are also implicitly describing the representation content of the dP_2 models)

$$\begin{aligned} W_{\mathrm{I}} = {} & h_1 \ \mathrm{tr}\left[X_{23}X_{35}X_{52}\right] - h_2 \ \mathrm{tr}\left[X_{23}Y_{31}X_{12}\right] - h_3 \ \mathrm{tr}\left[Y_{43}X_{35}X_{54}\right] + \\ & + h_4 \ \mathrm{tr}\left[X_{43}Y_{31}X_{15}X_{54}\right] + h_5 \ \mathrm{tr}\left[Y_{43}X_{31}X_{12}X_{24}\right] - \\ & - h_6 \ \mathrm{tr}\left[X_{52}X_{24}X_{43}X_{31}X_{15}\right], \end{aligned}$$

[9] In the D3-brane theory at the singularity, baryonic operators $\mathcal{O}$ on the D3-brane worldvolume couple to the boundaries of corresponding wrapped D3-branes via a term $\int ds\, \mathcal{O} \cdot \int_{C_i} \sqrt{g}$, where C_i is a (possibly vanishing) three-cycle. So a vev for $\mathcal{O}$ acts as a source for $\mathrm{Vol}(C_i)$.

$$\begin{aligned}W_{\rm II} = \; & h_1 \,\mathrm{tr}\left[Y_{41}X_{15}X_{54}\right] - h_2 \,\mathrm{tr}\left[Y_{41}Y_{13}X_{34}\right] + h_3 \,\mathrm{tr}\left[X_{21}Y_{15}X_{52}\right] - \\ & - h_4 \,\mathrm{tr}\left[X_{21}X_{13}X_{32}\right] + h_5 \,\mathrm{tr}\left[Z_{41}X_{13}X_{34}\right] - h_6 \,\mathrm{tr}\left[X_{41}Y_{15}X_{54}\right] + \\ & + h_7 \,\mathrm{tr}\left[X_{41}Y_{13}X_{32}X_{24}\right] - h_8 \,\mathrm{tr}\left[Z_{41}X_{15}X_{52}X_{24}\right] . \end{aligned} \tag{5.1}$$

Now, in dP_3 Model I, if we turn on a vev for any chiral superfield of $\mathcal{R}$-charge 1/3, it is easy to see that the resulting low-energy effective theory describing the cone over dP_2 is the first dP_2 model. Similarly, in dP_3 Model IV, turning on a vev for any of the chiral fields of $\mathcal{R}$-charge 1/3 results in the second dP_2 model. These blow-ups from $\mathcal{N}=1$ dual dP_3 models imply that the dP_2 models are also related by $\mathcal{N}=1$ duality. One can also see this directly by applying the $\mathcal{N}=1$ duality to SU(N) factor '1' or '4' in the first dP_2 model.

We also observe that in the dP_3 Models II and III (for which the discrete D_{12} is broken to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and for which there are now two orbits of baryons of $\mathcal{R}$-charge $\frac{1}{3}N$), we can turn on vevs of $\mathcal{R}$-charge $\frac{1}{3}N$ operators to realize either of the two dP_2 models from each dP_3 model. In dP_3 Model II, vevs for the baryons X_{14}^N, X_{34}^N, X_{53}^N, or X_{51}^N yield the first dP_2 model, while vevs for X_{42}^N and X_{65}^N yield the second dP_2 model. In dP_3 Model III, vevs for the baryons also X_{14}^N, X_{34}^N, X_{15}^N, or X_{35}^N also yield the first dP_2 model, while vevs for the baryons X_{52}^N or X_{42}^N yield the second dP_2 model.

We can also realize the pair of $\mathbb{F}_0$ models from the dP_3 models. In this case, the cone over $\mathbb{F}_0$ is obtained from the cone over dP_3 by blowing-up any of the three pairs of lines (L_1, L_4), (L_2, L_5), or (L_3, L_6). So the baryon/brane correspondence in the dP_3 models implies that turning on vevs for the corresponding pairs of $\mathcal{R}$-charge $\frac{1}{3}N$ baryons will realize these $\mathbb{F}_0$ models. For completeness, we record here the two superpotentials of the $\mathbb{F}_0$ models (the first of which appears in [5], both of which appear in [15]):

$$\begin{aligned}W_{\rm I} = \; & h_1 \,\mathrm{tr}\left[X_{14}X_{42}Y_{23}Y_{31}\right] + h_2 \,\mathrm{tr}\left[Y_{14}Y_{42}X_{23}X_{31}\right] - \\ & - h_3 \,\mathrm{tr}\left[Y_{14}X_{42}X_{23}Y_{31}\right] - \mathrm{tr}\left[X_{14}Y_{42}Y_{23}X_{31}\right] , \\ W_{\rm II} = \; & h_1 \,\mathrm{tr}\left[x_{21}y_{14}y_{42}\right] + h_2 \,\mathrm{tr}\left[y_{21}y_{13}x_{32}\right] + h_3 \,\mathrm{tr}\left[w_{21}x_{13}y_{32}\right] + h_4 \,\mathrm{tr}\left[z_{21}x_{14}x_{42}\right] - \\ & - h_5 \,\mathrm{tr}\left[w_{21}y_{14}x_{42}\right] - h_6 \,\mathrm{tr}\left[y_{21}x_{14}y_{42}\right] - h_7 \,\mathrm{tr}\left[x_{21}y_{13}y_{32}\right] - h_8 \,\mathrm{tr}\left[z_{21}x_{13}x_{32}\right] . \end{aligned} \tag{5.2}$$

We can easily check that the first $\mathbb{F}_0$ model arises from the dP_3 Model I, and the second $\mathbb{F}_0$ model arises from dP_3 Model IV. (Of course, we could also consider the $\mathbb{F}_0$ models as arising from deformations of the corresponding dP_2 models as well.) So again the two $\mathbb{F}_0$ models are related by $\mathcal{N}=1$ duality. In fact, applying a $\mathcal{N}=1$ duality transformation to any SU(N) factor of the gauge group of the first $\mathbb{F}_0$ model results in the second $\mathbb{F}_0$ model.

6. Conclusion

The fact that all three of the known examples of toric duality really arise from Seiberg's $\mathcal{N}=1$ duality leads us to conjecture that this result is general, for all examples of toric duality (and surely there are more) arising from partial resolutions of abelian orbifolds. If

JHEP12(2001)001

such a conjecture is true, it reassures us of the validity of the general procedures used to derive the worldvolume gauge theory descriptions of D3-branes at singularities which arise as partial resolutions of abelian orbifolds.

The gauge theories we encounter in our discussion are interesting nontrivial examples of dual theories, exhibiting the features of dangerously irrelevant operators, quantum modifications of chiral rings, and RG trajectories that do not come near free fixed points. It would be interesting to pursue the study of these theories further, along the lines of [12], establishing the precise operator mapping (with normalizations and scales). In particular, the structure of the quantum corrected chiral ring deserves closer study. There should be a deformation of this corresponding to reducing the coupling constant such that a limiting form reproduces the classical ring, at least in the case of an asymptotically free theory such as Model I.

It would also be interesting to understand better the holographic aspects of these systems. We understand how the conformal field theory is dual to an AdS compactification. Does the dynamics of the gauge theory at intermediate scales (below the Higgs scale but above the masses of lightest massive states) have a useful interpretation in terms of the worldvolume theory or the holographic dual? For instance, the duality cascade of [26] is an example of very interesting physics at these scales. We leave such questions for further work.

Acknowledgments

The authors would like to thank Brian Greene and Calin Lazaroiu for their collaboration in the earlier stages of this project. We would also like to thank David Morrison for helpful conversations. C.E.B. would like to thank Aaron Bergman and Chris Herzog for discussions of their volume computations on Einstein-Sasaki fivefolds, and he would like to thank Igor Klebanov for pointing out the relevance of those computations to this project. The work of C.E.B. is supported by an NSF Graduate Research Fellowship and under NSF grant PHY-9802484. M.R.P. would like to acknowledge the hospitality of the Institute for Theoretical Physics, Santa Barbara, where part of this work was completed during the Duality Workshop, and thank the organizers of the workshop. In particular, a conversation with M. Strassler was helpful in encouraging us to complete the project. The work of M.R.P. is supported in part by NSF grant DMS-0074072.

A. Chiral operators in Model I

The ring of mesonic operators is Model I is generated by traces of (powers of) the following seven adjoint chiral operators in Model I (the equalities below follow from the F-term equations[10]):

$$\mathcal{M}_1 = X_{15}X_{53}X_{34}X_{41} = X_{42}X_{26}X_{65}X_{53}X_{34} \,,$$

[10] We have assumed that the couplings h_i appearing in W are real and positive — otherwise, a field redefinition may be used to realize this situation.

$$
\begin{aligned}
\mathcal{M}_2 &= X_{65}X_{53}X_{32}X_{26} = X_{65}X_{53}X_{34}X_{41}X_{16}\,,\\
\mathcal{M}_3 &= X_{54}X_{41}X_{16}X_{65} = X_{53}X_{32}X_{21}X_{16}X_{65}\,,\\
\mathcal{M}_4 &= X_{16}X_{63}X_{32}X_{21} = X_{16}X_{65}X_{54}X_{42}X_{21}\,,\\
\mathcal{M}_5 &= X_{54}X_{42}X_{21}X_{15} = X_{16}X_{63}X_{34}X_{42}X_{21}\,,\\
\mathcal{M}_6 &= X_{34}X_{42}X_{26}X_{63} = X_{34}X_{42}X_{21}X_{15}X_{53}\,,\\
\mathcal{M}_7 &= X_{32}X_{26}X_{63} = X_{41}X_{15}X_{54} = X_{41}X_{16}X_{63}X_{34} = X_{21}X_{15}X_{53}X_{32}\\
&= X_{54}X_{42}X_{26}X_{65} = X_{34}X_{42}X_{21}X_{16}X_{65}X_{53}\,.
\end{aligned}
\tag{A.1}
$$

The classical baryonic generators in Model I of $\mathcal{R}$-charge $\frac{1}{3}N$ are

$$
\mathcal{B}_1 = X_{21}^N\,,\quad \mathcal{B}_2 = X_{65}^N\,,\quad \mathcal{B}_3 = X_{34}^N\,,\quad \mathcal{B}_4 = X_{53}^N\,,\quad \mathcal{B}_5 = X_{42}^N\,,\quad \mathcal{B}_6 = X_{16}^N\,. \tag{A.2}
$$

The classical baryonic generators in Model I of $\mathcal{R}$-charge $\frac{2}{3}N$ are

$$
\begin{aligned}
\mathcal{B}_1(s) &= X_{32}^s(X_{34}X_{42})^{N-s}\,, & \mathcal{B}_2(s) &= X_{26}^s(X_{21}X_{16})^{N-s}\,, & \mathcal{B}_3(s) &= X_{63}^s(X_{65}X_{53})^{N-s}\\
\mathcal{B}'_1(s) &= X_{15}^s(X_{16}X_{65})^{N-s}\,, & \mathcal{B}'_2(s) &= X_{54}^s(X_{53}X_{34})^{N-s}\,, & \mathcal{B}'_3(s) &= X_{41}^s(X_{42}X_{21})^{N-s}\,.
\end{aligned}
\tag{A.3}
$$

The classical baryonic generators in Model I of $\mathcal{R}$-charge N are

$$
\begin{aligned}
\mathcal{B}_1(s,t) &= (X_{65}X_{54})^s(X_{63}X_{34})^{t-s}(X_{65}X_{53}X_{34})^{N-t}\,,\\
\mathcal{B}'_1(s,t) &= (X_{34}X_{41})^s(X_{32}X_{21})^{t-s}(X_{34}X_{42}X_{21})^{N-t}\,,\\
\mathcal{B}''_1(s,t) &= (X_{21}X_{15})^s(X_{26}X_{65})^{t-s}(X_{21}X_{16}X_{65})^{N-t}\,,\\
\mathcal{B}_2(s,t) &= (X_{41}X_{16})^s(X_{42}X_{26})^{t-s}(X_{42}X_{21}X_{16})^{N-t}\,,\\
\mathcal{B}'_2(s,t) &= (X_{54}X_{42})^s(X_{53}X_{32})^{t-s}(X_{53}X_{34}X_{42})^{N-t}\,,\\
\mathcal{B}''_2(s,t) &= (X_{15}X_{53})^s(X_{16}X_{63})^{t-s}(X_{16}X_{65}X_{53})^{N-t}\,.
\end{aligned}
\tag{A.4}
$$

B. Chiral operators in Model II

Similarly for Model II, we have mesonic generators

$$
\begin{aligned}
\mathcal{M}_1 &= X_{12}X_{26}Y_{61} = X_{65}X_{53}X_{34}X_{42}X_{26}\,,\\
\mathcal{M}_2 &= X_{53}X_{32}X_{26}X_{65} = X_{51}X_{12}X_{26}X_{65} = X_{53}X_{34}Y_{46}X_{65}\,,\\
\mathcal{M}_3 &= X_{53}X_{32}Y_{26}X_{65} = X_{51}X_{12}Y_{26}X_{65} = X_{51}X_{14}Y_{46}X_{65}\,,\\
\mathcal{M}_4 &= X_{63}X_{32}Y_{26} = X_{65}X_{51}X_{14}X_{42}Y_{26}\,,\\
\mathcal{M}_5 &= X_{34}X_{42}Y_{26}X_{63} = X_{14}X_{42}X_{25}X_{51} = X_{14}X_{42}Y_{26}Y_{61}\,,\\
\mathcal{M}_6 &= X_{34}X_{42}X_{26}X_{63} = X_{14}X_{42}X_{26}Y_{61} = X_{34}X_{42}X_{25}X_{53}\,,\\
\mathcal{M}_7 &= Y_{46}Y_{61}X_{14} = X_{25}X_{51}X_{12} = Y_{46}X_{63}X_{34} = X_{25}X_{53}X_{32}\,,\\
&= X_{32}X_{26}X_{63} = X_{12}Y_{26}Y_{61}\,,\\
&= X_{34}X_{42}Y_{26}X_{65}X_{53} = X_{14}X_{42}X_{26}X_{65}X_{51}\,.
\end{aligned}
\tag{B.1}
$$

The classical baryonic generators in Model II of $\mathcal{R}$-charge $\frac{1}{3}N$ are

$$
\mathcal{B}_1 = X_{14}^N\,,\quad \mathcal{B}_2 = X_{65}^N\,,\quad \mathcal{B}_3 = X_{34}^N\,,\quad \mathcal{B}_4 = X_{53}^N\,,\quad \mathcal{B}_5 = X_{42}^N\,,\quad \mathcal{B}_6 = X_{51}^N\,. \tag{B.2}
$$

The classical baryonic generators in Model II of $\mathcal{R}$-charge $\frac{2}{3}N$ are

$$\begin{aligned} \mathcal{B}_1(s) &= X_{32}^s(X_{34}X_{42})^{N-s}, & \mathcal{B}_2(s) &= X_{26}^s Y_{26}^{N-s}, & \mathcal{B}_3(s) &= X_{63}^s(X_{65}X_{53})^{N-s}, \\ \mathcal{B}'_1(s) &= Y_{61}^s(X_{65}X_{51})^{N-s}, & \mathcal{B}'_2(s) &= (X_{51}X_{14})^s(X_{53}X_{34})^{N-s}, & \mathcal{B}'_3(s) &= X_{12}^s(X_{14}X_{42})^{N-s}. \end{aligned} \tag{B.3}$$

The classical baryonic generators in Model II of $\mathcal{R}$-charge N are

$$\begin{aligned} \mathcal{B}_1(s,t) &= (X_{65}X_{51}X_{14})^s(Y_{61}X_{14})^{t-s}(X_{65}X_{53}X_{34})^{N-t}\,, \\ \mathcal{B}'_1(s,t) &= (X_{34}X_{12})^s(X_{32}X_{14})^{t-s}((X_{34}X_{42})X_{14})^{N-t}\,, \\ \mathcal{B}''_1(s,t) &= X_{25}^s(X_{26}X_{65})^{t-s}(Y_{26}X_{65})^{N-t}\,, \\ \mathcal{B}_2(s,t) &= Y_{46}^s(X_{42}X_{26})^{t-s}(X_{42}Y_{26})^{N-t}\,, \\ \mathcal{B}'_2(s,t) &= (X_{51}X_{14}X_{42})^s(X_{51}X_{12})^{t-s}(X_{53}X_{34}X_{42})^{N-t}\,, \\ \mathcal{B}''_2(s,t) &= (Y_{61}X_{53})^s(X_{51}X_{63})^{t-s}(X_{51}(X_{65}X_{53}))^{N-t}\,. \end{aligned} \tag{B.4}$$

C. Chiral operators in Model III

Similarly for Model III, we have mesonic generators

$$\begin{aligned} \mathcal{M}_1 &= X_{26}Y_{61}X_{15}X_{52} = X_{26}Y_{63}X_{34}X_{42}\,, \\ \mathcal{M}_2 &= X_{26}Y_{63}X_{35}X_{52} = X_{26}X_{61}X_{15}X_{52} = Y_{63}X_{34}Y_{46}\,, \\ \mathcal{M}_3 &= Y_{26}X_{61}X_{15}X_{52} = Y_{26}Y_{63}X_{35}X_{52} = X_{61}X_{14}Y_{46}\,, \\ \mathcal{M}_4 &= Y_{26}X_{61}X_{14}X_{42} = Y_{26}X_{63}X_{35}X_{52}\,, \\ \mathcal{M}_5 &= Y_{26}X_{63}X_{34}X_{42} = Y_{26}Y_{61}X_{14}X_{42} = X_{63}X_{35}X_{56}\,, \\ \mathcal{M}_6 &= X_{26}Y_{61}X_{14}X_{42} = X_{26}X_{63}X_{34}X_{42} = Y_{61}X_{15}X_{56}\,, \\ \mathcal{M}_7 &= X_{56}X_{61}X_{15} = X_{56}Y_{63}X_{35} = Y_{46}Y_{61}X_{14} = Y_{46}X_{63}X_{34}\,, \\ &= X_{42}Y_{26}Y_{63}X_{34} = X_{52}Y_{26}Y_{61}X_{15} = X_{52}X_{26}X_{63}X_{35} = X_{42}X_{26}X_{61}X_{14}\,. \end{aligned} \tag{C.1}$$

The classical baryonic generators in Model III of $\mathcal{R}$-charge $\frac{1}{3}N$ are

$$\mathcal{B}_1 = X_{14}^N\,, \quad \mathcal{B}_2 = X_{52}^N\,, \quad \mathcal{B}_3 = X_{34}^N\,, \quad \mathcal{B}_4 = X_{15}^N\,, \quad \mathcal{B}_4 = X_{42}^N\,, \quad \mathcal{B}_6 = X_{35}^N\,. \tag{C.2}$$

The classical baryonic generators in Model III of $\mathcal{R}$-charge $\frac{2}{3}N$ are

$$\begin{aligned} \mathcal{B}_1(s) &= (X_{35}X_{52})^s(X_{34}X_{42})^{N-s}\,, \\ \mathcal{B}_2(s) &= X_{26}^s Y_{26}^{N-s}\,, \\ \mathcal{B}_3(s) &= X_{63}^s Y_{63}^{N-s}\,, \\ \mathcal{B}'_1(s) &= Y_{61}^s X_{61}^{N-s}\,, \\ \mathcal{B}'_2(s) &= (X_{35}X_{14})^s(X_{15}X_{34})^{N-s}\,, \\ \mathcal{B}'_3(s) &= (X_{15}X_{52})^s(X_{14}X_{42})^{N-s}\,. \end{aligned} \tag{C.3}$$

The classical baryonic generators in Model III of $\mathcal{R}$-charge N are

$$\mathcal{B}_1(s,t) = (X_{61}X_{14})^s(Y_{61}X_{14})^{t-s}(Y_{63}X_{34})^{N-t}\,,$$

$$\begin{aligned}
\mathcal{B}_1'(s,t) &= (X_{34}(X_{15}X_{52}))^s((X_{35}X_{52})X_{14})^{t-s}((X_{34}X_{42})X_{14})^{N-t}\,,\\
\mathcal{B}_1''(s,t) &= X_{56}^s(X_{52}X_{26})^{t-s}(X_{52}Y_{26})^{N-t}\,,\\
\mathcal{B}_2(s,t) &= Y_{46}^s(X_{42}X_{26})^{t-s}(X_{42}Y_{26})^{N-t}\,,\\
\mathcal{B}_2'(s,t) &= (X_{35}(X_{14}X_{42}))^s(X_{15}(X_{35}X_{52}))^{t-s}(X_{15}(X_{34}X_{42}))^{N-t}\,,\\
\mathcal{B}_2''(s,t) &= (Y_{61}X_{15})^s(X_{63}X_{35})^{t-s}(X_{61}X_{15})^{N-t}\,.
\end{aligned} \tag{C.4}$$

D. Chiral operators in Model IV

Similarly for Model IV, we have mesonic generators

$$\begin{aligned}
\mathcal{M}_1 &= Y_{56}X_{62}X_{25} = Y_{56}Y_{61}X_{15} = X_{46}X_{62}X_{24} = X_{46}Y_{63}X_{34}\,,\\
\mathcal{M}_2 &= Y_{56}X_{61}X_{15} = Y_{56}Y_{63}X_{35} = Y_{46}Y_{63}X_{34} = Y_{46}X_{62}X_{24}\,,\\
\mathcal{M}_3 &= Z_{56}X_{61}X_{15} = Z_{56}Y_{63}X_{35} = Y_{46}X_{61}X_{14} = Y_{46}Y_{62}X_{24}\,,\\
\mathcal{M}_4 &= Z_{56}X_{63}X_{35} = Z_{56}Y_{62}X_{25} = Z_{46}Y_{62}X_{24} = Z_{46}X_{61}X_{14}\,,\\
\mathcal{M}_5 &= X_{56}Y_{62}X_{25} = X_{56}X_{63}X_{35} = Z_{46}X_{63}X_{34} = Z_{46}Y_{61}X_{14}\,,\\
\mathcal{M}_6 &= X_{56}X_{62}X_{25} = X_{56}Y_{61}X_{15} = X_{46}Y_{61}X_{14} = X_{46}X_{63}X_{34}\,,\\
\mathcal{M}_7 &= Y_{56}X_{63}X_{35} = Y_{56}Y_{62}X_{25} = Y_{46}Y_{61}X_{14} = Y_{46}X_{63}X_{34}\,,\\
&= X_{56}X_{61}X_{15} = X_{56}Y_{63}X_{35} = X_{46}Y_{62}X_{24} = X_{46}X_{61}X_{14}\,,\\
&= Z_{56}X_{62}X_{25} = Z_{56}Y_{61}X_{15} = Z_{46}Y_{63}X_{34} = Z_{46}X_{62}X_{24}\,.
\end{aligned} \tag{D.1}$$

The classical baryonic generators in Model IV of $\mathcal{R}$-charge $\frac{1}{3}N$ are

$$\mathcal{B}_1 = X_{14}^N\,, \qquad \mathcal{B}_2 = X_{24}^N\,, \qquad \mathcal{B}_3 = X_{34}^N\,, \qquad \mathcal{B}_4 = X_{15}^N\,, \qquad \mathcal{B}_5 = X_{25}^N\,, \qquad \mathcal{B}_6 = X_{35}^N. \tag{D.2}$$

The classical baryonic generators in Model IV of $\mathcal{R}$-charge $\frac{2}{3}N$ are

$$\begin{aligned}
\mathcal{B}_1(s) &= (X_{35}X_{24})^s(X_{34}X_{25})^{N-s}\,,\\
\mathcal{B}_2(s) &= X_{62}^sY_{62}^{N-s}\,,\\
\mathcal{B}_3(s) &= X_{63}^sY_{63}^{N-s}\,,\\
\mathcal{B}_1'(s) &= Y_{61}^sX_{61}^{N-s}\,,\\
\mathcal{B}_2'(s) &= (X_{35}X_{14})^s(X_{15}X_{34})^{N-s}\,,\\
\mathcal{B}_3'(s) &= (X_{15}X_{24})^s(X_{14}X_{25})^{N-s}\,.
\end{aligned} \tag{D.3}$$

The classical baryonic generators in Model IV of $\mathcal{R}$-charge N are

$$\begin{aligned}
\mathcal{B}_1(s,t) &= (Y_{62}X_{24})^s(Y_{61}X_{14})^{t-s}(X_{62}X_{24})^{N-t}\,,\\
\mathcal{B}_1'(s,t) &= (X_{34}X_{15}X_{24})^s(X_{35}X_{24}X_{14})^{t-s}(X_{34}X_{25}X_{14})^{N-t}\,,\\
\mathcal{B}_1''(s,t) &= X_{56}^sY_{56}^{t-s}Z_{56}^{N-t}\,,\\
\mathcal{B}_2(s,t) &= Y_{46}^sX_{46}^{t-s}Z_{46}^{N-t}\,,\\
\mathcal{B}_2'(s,t) &= (X_{35}X_{14}X_{25})^s(X_{15}X_{35}X_{24})^{t-s}(X_{15}X_{34}X_{25})^{N-t}\,,\\
\mathcal{B}_2''(s,t) &= (X_{62}X_{25})^s(Y_{62}X_{25})^{t-s}(X_{61}X_{15})^{N-t}\,.
\end{aligned} \tag{D.4}$$

References

[1] See O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111].

[2] B.R. Greene, *D-brane topology changing transitions*, *Nucl. Phys.* **B 525** (1998) 284 [hep-th/9711124].

[3] M.R. Douglas, *D-branes, categories and $N = 1$ supersymmetry*, hep-th/0011017.

[4] M.R. Douglas and G.W. Moore, *D-branes, quivers and ALE instantons*, hep-th/9603167.

[5] D.R. Morrison and M.R. Plesser, *Non-spherical horizons, 1*, *Adv. Theor. Math. Phys.* **3** (1999) 1 [hep-th/9810201].

[6] M.R. Douglas, B.R. Greene and D.R. Morrison, *Orbifold resolution by D-branes*, *Nucl. Phys.* **B 506** (1997) 84 [hep-th/9704151].

[7] A. Kehagias, *New type-IIB vacua and their F-theory interpretation*, *Phys. Lett.* **B 435** (1998) 337 [hep-th/9805131].

[8] C.E. Beasley, *Superconformal theories from branes at singularities*, Duke University senior honors thesis, 1999.

[9] C. Beasley, B.R. Greene, C.I. Lazaroiu and M.R. Plesser, *D3-branes on partial resolutions of abelian quotient singularities of Calabi-Yau threefolds*, *Nucl. Phys.* **B 566** (2000) 599 [hep-th/9907186].

[10] B. Feng, A. Hanany and Y.-H. He, *D-brane gauge theories from toric singularities and toric duality*, *Nucl. Phys.* **B 595** (2001) 165 [hep-th/0003085].

[11] N. Seiberg, *Electric-magnetic duality in supersymmetric nonabelian gauge theories*, *Nucl. Phys.* **B 435** (1995) 129 [hep-th/9411149].

[12] D. Kutasov, A. Schwimmer and N. Seiberg, *Chiral rings, singularity theory and electric-magnetic duality*, *Nucl. Phys.* **B 459** (1996) 455 [hep-th/9510222].

[13] G. Tian and S.T. Yau, *Kähler-Einstein metrics on complex surfaces with $c_1 > 0$*, *Comm. Math. Phys.* **112** (1987) 175.

[14] G. Tian, *On Calabi's conjecture for complex surfaces of positive first Chern class*, *Invent. Math.* **101** (1990) 101.

[15] B. Feng, A. Hanany and Y.-H. He, *Phase structure of D-brane gauge theories and toric duality*, *J. High Energy Phys.* **08** (2001) 040 [hep-th/0104259].

[16] I.R. Klebanov and E. Witten, *Superconformal field theory on threebranes at a Calabi-Yau singularity*, *Nucl. Phys.* **B 536** (1998) 199 [hep-th/9807080].

[17] A. Bergman and C.P. Herzog, *The volume of some non-spherical horizons and the AdS/CFT correspondence*, hep-th/0108020.

[18] D. Anselmi, D.Z. Freedman, M.T. Grisaru and A.A. Johansen, *Nonperturbative formulas for central functions of supersymmetric gauge theories*, *Nucl. Phys.* **B 526** (1998) 543 [hep-th/9708042].

[19] S.S. Gubser, *Einstein manifolds and conformal field theories*, *Phys. Rev.* **D 59** (1999) 025006 [hep-th/9807164].

[20] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, 1978.

[21] R. Hartshorne, *Algebraic geometry*, Springer, 1977.

[22] W. Fulton, *Introduction to toric varieties*, Princeton Univ. Press, 1993.

[23] R. Bott and L. Tu, *Differential forms in algebraic topology*, Springer, 1982.

[24] C.P. Boyer and K. Galicki, *On Sasakian-Einstein geometry*, *Int. J. Math.* **11** (2000) 873, [math.DG/9811098].

[25] E. Witten, *Baryons and branes in anti de Sitter space*, *J. High Energy Phys.* **07** (1998) 006 [hep-th/9805112].

[26] I.R. Klebanov and M.J. Strassler, *Supergravity and a confining gauge theory: duality cascades and χSB-resolution of naked singularities*, *J. High Energy Phys.* **08** (2000) 052 [hep-th/0007191].